WORKING PAPER
Büro des Fiskalrates

The FISK OLG model - A numerical overlapping generations model for Austria

Model description v2.3

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Working Paper 11 / April 2025, Wien.

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A Numerical Overlapping Generations Model for Austria

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Vienna, April 29, 2025

Abstract

This paper presents an update of the technical description of the FISK OLG Model, a numerical overlapping generations (OLG) model of the Auerbach-Kotlikoff type, specifically developed for Austria. The FISK OLG model is designed to quantify the medium- and long-term effects of demographic changes and structural reforms. Household and firm decisions are microfounded, and the model operates dynamically in general equilibrium. Particular attention is given to a detailed representation of government revenues and expenditures. An extension of the model incorporates endogenous energy consumption and greenhouse gas emissions.

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Acknowledgments: The model invaluably profited from discussions and related work with Alexia Fürnkranz-Prskawetz and Miguel Sánchez-Romero during my stay at the Technical University of Vienna. I am highly indebted to Christian Keuschnigg for providing advice and his technical notes on CGE modeling. I want to thank Susanne Maidorn who prepared the life-cycle profiles from EU-SILC data and Bernhard Hammer for providing NTA profiles data electronically. I am particularly thankful to Zuzana Molnárová for carefully reading this documentation and helping to improve it. Special thanks go to Johannes Berger, Thomas Davoine, Gerhard Fenz, Susanne Forstner, Johannes Holler, Markus Knell, Reinhard Koman, Paul Pichler, Christian Ragac, Lukas Reiss and Ludwig Strohner for valuable discussions, comments and inputs.

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1 Introduction

The aim of the dynamic model of Auerbach-Kotlikoff-type is to address fiscal policy questions that are particularly relevant in the medium and long run. The model is designed and calibrated such that it captures the economic environment and institutional features specific to Austria. As demographic change is an important determinant of future fiscal developments, we put particular emphasis on a detailed representation of the population structure. Persons differ in the following dimensions: age (recorded in single years in the tradition of Auerbach and Kotlikoff, 1987), birth year, highest attained education (primary, secondary, or tertiary), and savings type (Ricardian "consumption smoother" or "hand-to-mouth" consumers following Campbell and Mankiw, 1989). We refer to a particular combination of characteristics as a cell. Over time, persons move between cells, although the transition is subject to some rather intuitive restrictions. Obviously, persons cannot change their birth year, and age increases by exactly one year each year. Further, we assume that persons cannot change their savings type or highest attained education. The demographic module differentiates persons by sex and provides information on the number of persons and the vital rates (fertility, mortality, and net migration) per cell. In the economic part of the model, the household sector is populated by a representative unisex household per cell. Persons under the age of 15 do not make economic decisions and are allocated to the adult population by adjusting household weights accordingly. The representative households make decisions along the following margins: consumption, labor market participation, retirement, and hours supply. Therefore, age- and education-specific participation, income, consumption, etc., profiles by cohort are model outcomes. Aggregating these model outcomes across cells results in the macro aggregates of the household sector. The only source of uncertainty for an individual agent is the time of death, which follows a stochastic process. Besides that, agents are gifted with perfect foresight, i.e., future events are expected by agents with certainty. However, the model allows for unanticipated shocks.

The model is dynamic and solved in general equilibrium, i.e., prices are the result of interaction between households, firms, the government, and the rest of the world in product, factor, and asset markets. Representative firms make forward-looking decisions concerning investment, labor demand, and price setting. Firms take production factors labor, capital, and public capital as inputs and turn them into outputs. The labor-augmenting technological process that determines firm productivity is assumed to be exogenous. The government affects the economy by influencing agents' resource constraints (via taxes and transfers) and by participating in product markets (via public consumption and public investment). Further, the government issues debt, which is an imperfect substitute for other asset types (domestic and foreign firm assets, and foreign public debt). Therefore, there is no arbitrage, and assets can earn different returns. Demand for different asset types is based on a portfolio optimization problem of households. Demand for goods is allocated between domestic and imported goods following the Armington (1969) assumption.

Particular emphasis was placed on capturing government revenues and expenditures. Most of the taxes are proportional, with the exception of a progressive income tax, which is based on non-linear tax functions. Demography-related expenditure is mostly modeled using age-skill-specific unit cost profiles. Special attention was given to the pension system. Pensions are based on persons' income histories and are subject to different pension system regimes (the old systems differentiating between private sector employees and civil servants, and the new harmonized pension account system). In contrast to many comparable models, we do not treat a (current) base year or base period as a steady state. The model is fit to the historical time series with an initial steady state dating many generations in the past. This allows us to capture important non-stationarities as observed in the data. Examples of such dynamic developments are the non-stationary relationships between the current population structure and current vital rates, or between the current primary balance and the current level of debt. Furthermore, this approach ensures that future trends, such as aging, are already included in agents' expectations. In addition, this approach allows us to incorporate historical reforms, such as pension reforms, with gradual, yet long-lasting consequences.

This paper contains a detailed documentation of the model. Note that the modeling approach is modular. Different specifications of, e.g., utility or production functions, can be used. Not all model features are necessarily included in every application, as some features are optional, and others are even mutually exclusive. Section 2 describes model assumptions, formulates agents' optimization problems and optimality conditions, and lists the model's accounting identities. While functional forms are kept general in Section 2, they are specified in Section 3. Section 4 gives a sketch of the numerical solution algorithm. Section 5 describes how we bring the model to the data. The appendix contains – among other things – a documentation of the parameterizations and model features used in selected past applications.

2 Model Description[1]

2.1 Description of the Economy

The economy is populated by overlapping generations of households with heterogeneous agents. There are $2 \cdot S + 1$ factor markets: $2 \cdot S$ types of labor (S levels of skill times two types of age-specific labor: "young" and "old") and capital, with prices w^j, $j \in \{1, \ldots, S\} \times \{Y, O\}$, and p^K. There is a continuum of product markets for differentiated domestic goods (Dixit and Stiglitz, 1977) that are assembled into a homogeneous final good sold at price p^h to domestic consumers, firms, and the government, as well as abroad. The domestic agents also demand a foreign good sold at price $p^m = 1$ (numéraire).[2] Households hold four types of assets: domestic and foreign government bonds, and domestic and foreign firm assets. The asset types are imperfect substitutes and can earn different return rates: $r^{G,h}$, $r^{G,m}$, $r^{V,h}$, and $r^{V,m}$. The model is expressed in real terms. The economy is modeled as being small, i.e., foreign prices and asset return rates are not influenced by the domestic economy and are therefore exogenous. The model variables are detrended by population growth, labor-augmenting productivity growth, and inflation.

2.2 Timing Convention

The model is set up in discrete time, which implies that we have to define a consistent timing convention[3], which we briefly describe in this section. All goods flows, such as labor income, consumption, and tax/transfer payments, occur at the beginning of a period. First, all demographic shocks, i.e., mortality, births, and migration, happen at the end of a period, implying a new population size at the beginning of the next period. Consequently, we measure population size at the beginning of the period. There are two important exceptions. Second, end-of-period assets are observed, which determine interest earnings. As all taxes occur at the beginning of the period, interest tax payments, which are based on end-of-period assets, have to

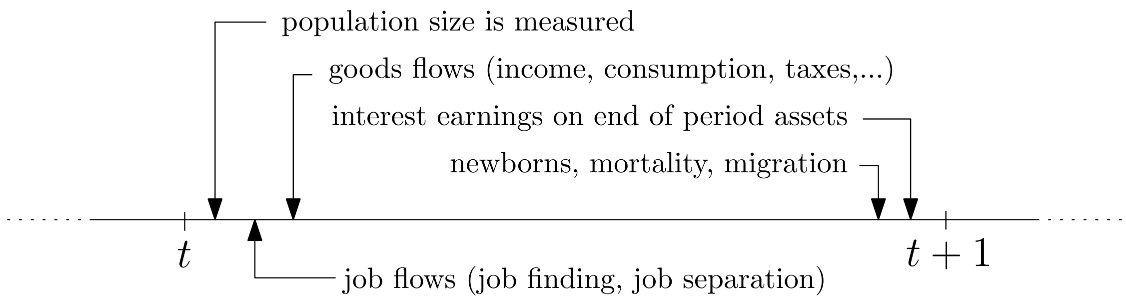
[1]At the beginning a note of caution is due. Because of the complexity of the model the task of presenting it in precise notation is not always easy, which is why precise notation is occasionally sacrificed for the sake of readability. This comes in form of casually dropping indices, multiple usage of the same (Greek) letter for different variables, etc., all of which should hopefully be clear from context.

[2]This means that p^h also reflects the terms of trade.

[3]The choice of the timing convention, in principle, has no real impact on the model if applied consistently.

be discounted by one period. All other tax or transfer flows are straightforward to compute. Figure 2.1 illustrates the timing convention.

Figure 2.1: Illustration of the model timing convention



2.3 Demography

The model distinguishes $A \times S \times 2$ representative households[4] that differ by age $a \in \mathcal{A} = \{0, 1, \dots, \overline{a} - 1, \overline{a}\} \subset \mathbb{N}$, skill[5] $s \in \mathcal{S} = \{1, \dots, S\}$ and savings type $k \in \mathcal{K} = \{U, C\}$, which can either be 'Ricardian' (or Unconstrained) or 'hand-to-mouth' (or Constrained). It is often convenient to index households by a single index $i \in \mathcal{A} \times \mathcal{S} \times \mathcal{K}$. The number of age groups is $A = \overline{a} + 1$. If $a \in \{\underline{a}, \dots, \overline{a}\}$, we speak of adults who make economic decisions. In contrast, age groups with $a \in \{0, \dots, \underline{a} - 1\}$ are children who do not make economic decisions. While persons progress in age, skill, and savings type are determined at birth and do not change during the lifespan. Aging is deterministic, i.e., a household of age a at time t is of age $a + n$ in $t + n$ (unless it died in the meantime). Indexing by time t is referred to as the 'period view', while indexation by $z = t - a$, i.e., by the year of birth, is referred to as the 'cohort view'. Let $\tilde{N}_t^{a,s,k}$ be the mass of households of age a, skill s, and savings type k at time t. A tilde marks a non-detrended variable.

[4]Households are interpreted in an individualistic, unisex way, i.e., we do not model actual household structures, except for a household size that captures the presence of economically inactive children. Hence, the terms 'household' and 'individual' are used interchangeably.

[5]The differentiation by skill can easily be generalized to let the model represent any kind of different subpopulation, e.g., differentiated by nationality, etc. To give an example: In order to address migration of a specific type of persons, e.g., non-EU citizens, one would form (at least) two subpopulations: Austrian population comprised of non-EU citizens and Austrian population comprised of all but non-EU citizens. This allows for differentiated socioeconomic characteristics of the households even conditional on age and skill in both subpopulations, as well as for different net migration in those two subpopulations. The solution mechanism of the model (see section 4) in principle allows for a high degree of heterogeneity, with computational time only increasing about linearly.

The total population is $\tilde{N}_t = \sum_{a=0}^{\overline{a}} \sum_{s=1}^{S} \sum_{k \in \{U,C\}} \tilde{N}_t^{a,s,k}$ or in short notation $\tilde{N}_t = \sum_{i \in \mathcal{A}\times\mathcal{S}\times\mathcal{K}} \tilde{N}_t^i$. Population growth is $\mathcal{N}_t = \tilde{N}_{t+1}/\tilde{N}_t = 1 + n_t$. For detrending the model by population growth, we divide all variables representing person counts by the total population. Hence, we define $N_t^i = \tilde{N}_t^i / \tilde{N}_t \overline{N}$, etc., where $\overline{N}$ is an arbitrary scaling factor. N_t^i therefore denotes the mass of households of characteristic i at time t in detrended terms. Household i dies at the end of period t at mortality rate $(1 - \gamma_t^i)$.[6] $NB_t^{s,k}$ is the number of newborns at the end of period t in detrended terms, and Mig_t^i is the exogenous net migration flow into the economy with age a and skill class s (again in detrended terms). We assume that migrants (conditional on age, skill, and savings type) are indistinguishable from natives and (as an approximation) that the foreign country is indistinguishable from the domestic economy. This allows us to incorporate migration in an extremely simple manner for two reasons. First, in this case, we do not have to keep track of when a person immigrated into the economy. Second, households do not have to form expectations about if and when they are hit by an emigration shock, and therefore, life-cycle optimization, as presented in section 2.6, is unaffected by incorporating migration. The demographic structure is defined by a simple system of difference equations.

$$N_{t+1}^{0,s,k} = NB_t^{s,k} + Mig_t^{0,s,k} \tag{1}$$

$$N_{t+1}^{a+1,s,k} = \gamma_t^{k,a,s} N_t^{a,s,k} / \mathcal{N}_t + Mig_t^{a+1,s,k}, \quad \gamma_t^{\overline{a},s,k} = 0 \ \forall t. \tag{2}$$

The restriction $\gamma^{k,\overline{a},s} = 0$ guarantees that the maximum attainable age is $\overline{a}$. Observe the important assumption that households remain in their skill class from the beginning to the rest of their lives, i.e., there are no cross-skill transitions. The total population size (in detrended terms) is

$$N_t = \sum_{a=0}^{\overline{a}} \sum_{s=1}^{S} \sum_{k \in \{U,C\}} N_t^{a,s,k} \quad \text{or} \quad N_t = \sum_{i \in \mathcal{A}\times\mathcal{S}\times\mathcal{K}} N_t^i, \tag{3}$$

Often also simply denoted $N_t = \sum_i N_t^i$.[7] Sub-aggregates are denoted and computed accordingly, e.g., $N_t^U = \sum_{a=0}^{\overline{a}} \sum_{s=1}^{S} N_t^{a,s,U}$, $N_t^{s=1} = \sum_{a=0}^{\overline{a}} \sum_{k \in \{U,C\}} N_t^{a,1,k}$, etc. Fur-

[6] In the implementation, mortality rates typically only differ by age and skill, and not by savings type, due to lack of data.

[7] Note that because of detrending, N_t will be constant and equal to the arbitrary scaling factor $\overline{N}$, while this is not the case for the age distribution, e.g., N_t^i/N_t.

ther, life expectancy of cohort z at age a and average age[8] of the population at time t per skill group are given as

$$\text{life expectancy:} \quad LE_z^{a,s,k} = \sum_{x=a}^{\bar{a}} x(1-\gamma_z^{x,s,k}) \prod_{j=a}^{x-1} \gamma_z^{j,s,k}.$$

$$\text{average age:} \quad avage_t = \frac{\sum_{a=0}^{\bar{a}} a N_t^{a,s,k}}{\sum_{a=0}^{\bar{a}} N_t^{a,s,k}}.$$

For the sake of modeling survivors' pensions, we partition each cohort into three family statuses: "single" (S), "married" (M), and "widowed" (W), with relative shares denoted by f, i.e., $f_t^{S,a} + f_t^{M,a} + f_t^{W,a} = 1$. Using some simplifying assumptions, in particular, that there are no differences between savings types or migration histories, and that a married couple always populates the same age-skill cell, one can write down this system of flows between the family statuses (dropping the skill index for easier readability):

$$\begin{aligned}
f_{t+1}^{S,a+1} &= f_t^{S,a}(1-\pi_t^{S,a}) + f_t^{M,a}\pi_t^{D,a}, \\
f_{t+1}^{M,a+1} &= f_t^{M,a}(\gamma_t^a - \pi_t^{D,a}) + f_t^{S,a}\pi_t^{S,a} + f_t^{W,a}\pi_t^{W,a}, \\
f_{t+1}^{W,a+1} &= f_t^{W,a}(1-\pi_t^{W,a}) + f_t^{M,a}(1-\gamma_t^a),
\end{aligned}$$

where π^S and π^W are the marriage rates for singles and widowed persons, respectively, and π^D is the divorce rate. Note that in the decision-making of the representative households/cohorts, we pool the incomes of households of different family statuses. This means that household decisions are independent of family status.

2.4 Employment Dynamics

We model unemployment following a simplified search and matching framework of the Diamond-Mortensen-Pissarides style (see e.g., Pissarides, 2000), where aggregate unemployment persists due to continuous flows into and out of employment, and job finding is a time-consuming process. Dropping indices for skill and age-labor type (see next section), let $LF_t^a = \delta_t^a N_t^a$ be the age-specific labor force, with δ_t^a being

[8] In case of no migration, there is a simple solution for average age in steady state: $\frac{\sum_{a=0}^{\bar{a}} a \prod_{j=0}^{a-1} \gamma^{j,s,k}}{1+\sum_{a=0}^{\bar{a}} \prod_{j=0}^{a-1} \gamma^{j,s,k}}$.

the participation rate. With unemployment and employment rates (conditional on participation) of u_t^a and e_t^a (with $u_t^a + e_t^a = 1$), the number of unemployed and employed by age are $U_t^a = u_t^a LF_t^a$ and $E_t^a = e_t^a LF_t^a$. We assume exogenous job separation rates χ_t^a. The job finding rate (from the perspective of the unemployed worker) is $q_t^{w,a}$, which will be discussed further below. The dynamics of the number of employed and unemployed, w.l.o.g., are

$$E_t^a = \left[E_{t-1}^{a-1} + \pi^e(dLF_t^a)\right](1-\chi_t^a) + \left[U_{t-1}^{a-1} + \pi^u(dLF_t^a)\right] q_t^{w,a}, \tag{4}$$

$$U_t^a = \left[U_{t-1}^{a-1} + \pi^u(dLF_t^a)\right](1-q_t^{w,a}) + \left[E_{t-1}^{a-1} + \pi^e(dLF_t^a)\right] \chi_t^a, \tag{5}$$

where $\pi^e(\cdot)$ and $\pi^u(\cdot)$ are functions capturing the effect of flows in and out of the labor force, with $dLF_t^a = LF_t^a - LF_{t-1}^{a-1}$. To considerably simplify the dynamics, we use the following assumption:

Assumption 2.1. *Proportionality assumption of flows in and out of the labor force: We assume that* $\pi^e(dLF_t^a) = e_{t-1}^{a-1} \cdot dLF_t^a$ *and* $\pi^u(dLF_t^a) = u_{t-1}^{a-1} \cdot dLF_t^a$.

Using this assumption of 'labor force re-basing'[9] allows us to write the dynamics in a simple form of rates.

$$e_t^a = e_{t-1}^{a-1}(1-\chi_t^a) + (1-e_{t-1}^{a-1}) q_t^{w,a}, \tag{6}$$

$$u_t^a = u_{t-1}^{a-1}(1-q_t^{w,a}) + (1-u_{t-1}^{a-1}) \chi_t^a. \tag{7}$$

We can now look at the job finding rate and the matching process in more detail. $q_t^{w,a} = s_t^a q(\theta_t)$ consists of two parts. s_t^a is search effort, a household choice, which is age-specific and proportionally increases the job finding rate. The second part, $q(\theta_t)$, is the matching probability per search effort unit, which increases with labor market tightness $\theta_t = \frac{V_t}{\bar{s}_t u_{t-1} LF_t} = \frac{v_t}{\bar{s}_t u_{t-1}}$, and is independent of age, i.e., firms cannot discriminate by age in their hiring decisions, and there is just a single labor market per skill and age-labor type. V_t denotes the total number of vacancies, and v_t the vacancy-to-labor force ratio. The total numbers of unemployed, employed, and participants are simply $U_t = \sum_a U_t^a$, $E_t = \sum_a E_t^a$, and $LF_t = \sum_a LF_t^a$. The total unemployment and employment rates (by skill and age-labor type) are $u_t = U_t/LF_t$ and $e_t = E_t/LF_t$, respectively. $\bar{s}_t = \left(\sum_a s_t^a u_{t-1}^{a-1} LF_t^a\right) / \left(\sum_a u_{t-1}^{a-1} LF_t^a\right)$ denotes average search effort. We assume a standard, linearly homogeneous matching function

[9] This assumption is admittedly more realistic for flows out of the labor force, e.g., because of death, emigration, or retirement, than for flows into the labor force.

$\mathcal{M}(\bar{s}_t u_{t-1} LF_t, V_t)$, which relates to the matching probability as follows:

$$q(\theta_t) \equiv \mathcal{M}(\bar{s}_t u_{t-1} LF_t, V_t)/(\bar{s}_t u_{t-1} LF_t) = \mathcal{M}(1, \theta_t) \tag{8}$$

Similarly, the job filling probability is $q^f(\theta_t) = q(\theta_t)/\theta_t$, and the law of motion for employment is

$$E_t = V_t q_t^f + e_{t-1} LF_t (1 - \chi_t), \tag{9}$$

where $\chi_t = (\sum_a \chi_t^a e_{t-1}^a LF_t^a)/(\sum_a e_{t-1}^a LF_t^a)$ is the average separation rate. For later reference, note that $\partial e_t^a / \partial s_t^a = -\partial u_t^a / \partial s_t^a = u_{t-1}^{a-1} q(\theta_t)$, and that $\partial E_t / \partial V_t = q_t^f$.

2.5 Technology, Price Inflation and Detrending

Technology Γ_t is labor-augmenting. Productivity growth is exogenously given and denoted by $\mathcal{G}_t = \Gamma_{t+1}/\Gamma_t$, with the growth rate $g_t = \mathcal{G}_t - 1$. In addition, the model is detrended by trend inflation. The current price level is $\Gamma_t^{€}$. Then, the growth factor and rate are: $\mathcal{G}_t^{€} = \Gamma_{t+1}^{€}/\Gamma_t^{€}$ and $g_t^{€} = \mathcal{G}_t^{€} - 1$. The value of a non-detrended quantity $\tilde{X}_t$ is $\tilde{p}_t^m \tilde{X}_t$, which is converted into its detrended counterpart as follows: $X_t = \tilde{p}_t^m / \Gamma_t^{€} \cdot \tilde{X}_t / \Gamma_t$.[10] Let $\tilde{X}_t$ be a non-detrended per-capita stock variable (in quantity terms) subject to the nominal interest rate i_t (end-of-period) and a net per-period inflow $\tilde{Z}_t$ (in quantity terms). The law of motion of the value of the stock variable in undetrended and detrended terms is given as follows:

$$\tilde{p}_{t+1}^m \tilde{X}_{t+1} = (1 + i_t) \left[\tilde{p}_t^m \tilde{X}_t + \tilde{p}_t^m \tilde{Z}_t \right] \quad \Rightarrow \quad \mathcal{G}_t X_{t+1} = (1 + r_t) \left[X_t + Z_t \right]. \tag{10}$$

The right-hand side follows from dividing by $\Gamma_t \Gamma_t^{€}$ and defining the real interest factor as $1 + r_t = (1 + i_t)/\mathcal{G}_t^{€}$, which gives a difference equation purely in detrended per-capita variables. At the aggregate level, we, in addition, detrend by population growth. For this, we define $\hat{\mathcal{G}}_t = \mathcal{G}_t \mathcal{N}_t$ and $\hat{g}_t = \hat{\mathcal{G}}_t - 1$.

[10] $\Gamma_t^{€}$ relates current and real price as follows: $\tilde{p}_t^m = \Gamma_t^{€} p_t^m$. Because $p_t^m = 1$, it is typically omitted when using detrended variables, and we simply write quantity X_t. All real prices occurring in the model, such as p^h, p^C, w, are measured in relative terms to p^m. As p^h and p^m are producer prices, the empirical counterpart of $g^{€}$ would be the domestic output deflator growth corrected by terms of trade.

2.6 Households

Households pass through two life stages: childhood and adulthood. Individuals of age $a \in \{0, \ldots, \underline{a} - 1\}$ do not make any economic decisions.[11] In the model, this implies, for example, that children have zero consumption (which is implicitly included in the consumption of the adults, even if parent-child pairs are not explicitly modeled), zero asset holdings, i.e., they are excluded from intervivo transfers, and they do not directly receive government transfers or pay taxes. Hence, all the decisions below apply only to households during adulthood. The model distinguishes (next to heterogeneity in age and skill) between two types of (adult) households: unconstrained and constrained households. Unconstrained households – or Ricardian households – can smooth consumption by saving and borrowing, i.e., they face an intertemporal budget constraint. In contrast, constrained households cannot save or borrow, which implies that their per-period consumption level is equal to their per-period disposable income, and their asset holdings are always 0. We also refer to these households as hand-to-mouth households. A share of π^s of $N^{a,s}$ is unconstrained. The share can vary by skill class and cohort z. However, once assigned at the beginning of life, a household cannot switch the characteristic of being constrained or unconstrained during its lifetime. To distinguish the savings types, we use the superscripts U for unconstrained and C for constrained households. Aggregation over savings types works as follows (using population size N, consumption C, and per-period income from labor and public transfers y as examples):

$$
\begin{aligned}
N_z^{U,a,s} &= \pi_z^s \cdot N_z^{a,s}, \\
N_z^{C,a,s} &= (1 - \pi_z^s) \cdot N_z^{a,s}, \\
N_t^{a,s} &= N_t^{U,a,s} + N_t^{C,a,s}, \\
C_t &= \sum_{a,s} C_t^{U,a,s} N_t^{U,a,s} + \sum_{a,s} C_t^{C,a,s} N_t^{C,a,s} = C_t^U + C_t^C, \\
y_t &= \sum_{a,s} y_t^{a,s} N_t^{a,s} = \sum_{a,s} y_t^{a,s} N_t^{U,a,s} + \sum_{a,s} y_t^{a,s} N_t^{C,a,s} = y_t^U + y_t^C.
\end{aligned}
$$

The next subsection describes the optimization problem of a representative household (a, s) for the unconstrained savings type. As there are no dynamics in skills,

[11] Children play a role in the model as they influence the age-dependent household size weights in household preferences (see section 3.1) and to capture age-specific public consumption (e.g., education costs) and family transfers. All household decision variables (consumption, labor supply, etc.) indexed $a \in \{0, 1, \ldots, \underline{a} - 1\}$ are 0 by assumption.

we simplify the notation and drop the s superscript when appropriate throughout this section.

2.6.1 Ricardian Households

Household income

Ricardian (or unconstrained) households can build up a stock of assets A^a. All households are assumed to have the same preferences concerning portfolio choice. Portfolio investment shares will be independent of the size of the individual asset stock, which implies that the asset allocation problem can be solved at the aggregate level (in section 2.9). We assume that asset holdings of households that died because of a mortality shock are bequeathed to younger age groups (of unconstrained households). Note that end-of-period asset holdings of households of maximum age $\overline{a}$ are zero because they die with certainty.[12] ab_t^a denotes the flow incomes of younger households from these accidental bequests. Unconstrained households face the following intertemporal budget constraint (11).

$$\mathcal{G}_t A_{t+1}^{a+1} = \bar{R}_t^W \left[A_t^a + \bar{y}_t^{U,a} - p_t^C C_t^{U,a} \right], \quad \bar{y}_t^{U,a} = y_t^a + iv_t^a + ab_t^a - Z_t^{U,a}, \tag{11}$$

where $\bar{y}_t^{U,a}$ denotes total per period income flows. iv^a are intervivo transfers between different generations, with the condition that $\sum_{a=\underline{a}}^{\overline{a}} iv_t^a N_t^{U,a} = 0$, i.e., iv_t^a can be positive for some and negative for other age groups. This is introduced in order to fit the model better to age-specific asset profiles in the implementation.[13] $Z_t^{U,a}$ denotes financial transfers abroad. $p_t^C C_t^{U,a}$ is consumption expenditure. Here, C represents composite consumption consisting of varieties of domestic and imported goods. p^C denotes the price of composite consumption per unit, including consumption taxes, while $\tilde{p}^C$ is the price without taxation, i.e., $p_t^C = (1+\tau_t^C)\tilde{p}_t^C$.[14] The effective after-tax interest factor for households, $\bar{R}^W$, is the weighted average of the after-tax interest factors of the different asset type holdings. Per period income (without intervivo

[12] Any household that survived to age $\overline{a}$ will consume all its per-period income and remaining assets in the last period.

[13] Note that in the presented model description, only unconstrained households give/receive intervivo transfers and accidental bequests. Except for giving accidental bequests, this could easily be relaxed to allow transfers also from and to constrained households.

[14] In an abuse of notation, the tilde here refers to before-tax prices and not to non-detrended values.

transfers and accidental bequests) y_t^a is given by

$$\text{income w/o priv. trans.:} \qquad y_t^a = \delta_t^a e_t^a (1-\tau_t^{W,a}) w_t^a \ell_t^a \theta_t^a + \delta_t^a u_t^a b_t^{u,a} + \delta_t^a b_t^{\delta,a} + (1-\delta_t^a) b_t^a - \tilde{p}_t^C \tau_t^{l,a}, \tag{12}$$

$$\text{non-part. income:} \qquad b_t^a = \phi_t^a b_t^{n,a} + (1-\phi_t^a)(1-\tau_t^{P,a}) y_{pens,t}^a, \tag{13}$$

$$\text{old-age pension income:} \qquad y_{oldage,t}^a = \left[\sum_{p \in \mathcal{P}} \alpha_t^{p,a} \varsigma_t^{p,a} P_t^{p,a} \right] + P_t^{0,a}, \tag{14}$$

$$\text{pension income:} \qquad y_{pens,t}^a = y_{oldage,t}^a + y_{surv,t}^a, \tag{15}$$

$$\text{survior income:} \qquad y_{surv,t}^a = \left[f_t^{w,a} \chi_t^w y_t^{w,a} + f_t^{o,a} \chi_t^o \right] / \upsilon_t^a, \tag{16}$$

$$\text{gross income base:} \qquad y_t^{b,a} = \delta_t^a \left[e_t^a \min \left\{ w_t^a \ell_t^a \theta_t^a, y_t^{cap} \right\} + u_t^a b_t^{p,a} \right]. \tag{17}$$

For many applications, it is convenient to define more structure on benefits $b_t^{u,a}$ and $b_t^{p,a}$, and on the survivors' pension income base $y_t^{w,a}$, particularly of the following form:

$$b_t^{u,a} = \phi_t^{u,a} \cdot \overline{(1-\tau_t^{W,a}) w_t^a \ell_t^a \theta_t^a}, \tag{18}$$

$$b_t^{p,a} = \phi_t^{p,a} \cdot \overline{w_t^a \ell_t^a \theta_t^a}, \tag{19}$$

$$y_t^{w,a} = \overline{\delta_t^a w_t^a \ell_t^a \theta_t^a} + \phi_t^{w,a} \overline{y_{pens,t}^a} \tag{20}$$

where the bar indicates average values, such that they are taken as given by the individual household.[15]

This requires some explanation. δ_t^a denotes the participation rate. Participation is modeled as a discrete choice (**extensive margin** of labor supply), which is then convexified, i.e., $\delta_t^a \in [0,1]$, using an income-pooling assumption[16] within a representative household of age a and skill s in order to keep the model tractable. Hence, δ_t^a is the share of household members of a representative household participating in the labor market. Conditional on participation, individuals can either be employed (e_t^a) or unemployed (u_t^a). The probability (or share of members in a household) of being employed is affected by the state of the labor market (taken as given from the individual perspective) and individual search effort s_t^a (**search margin** of labor supply). ℓ_t^a denotes the chosen number of hours of labor supply (**intensive margin**

[15]Ex-post, we then have $\overline{w_t^a \ell_t^a \theta_t^a} = w_t^a \ell_t^a \theta_t^a$, etc.

[16]See, for example, Andolfatto (1996).

of labor supply). w_t^a is the wage rate for one effective unit of labor input, and θ_t^a is an exogenous productivity parameter depending on age in order to replicate realistic wage-income-age profiles. $\tau_t^{W,a}$ is the total average tax rate on labor income, consisting of pension contributions, other contributions, and income tax. The tax rate can either be exogenously given (derived from microdata conditional on age and skill) or computed using a tax function and therefore solely determined by labor income (see Section 3.4). $b_t^{\delta,a}$ is a benefit conditional on participation (e.g., sick leave cash benefits). During unemployment, households receive benefits $b_t^{u,a}$, while during non-participation, households get transfer payments b^a. The variable ϕ_t^a is an exogenous policy parameter that indicates whether or not a household is eligible for pension payments $y_{pens,t}^a$. Otherwise, a household could receive non-participation transfers $b_t^{n,a}$ net of taxes. The retirement share is $\upsilon_t^a = (1-\delta_t^a)(1-\phi_t^a)$, where $\phi_t^a \in [0,1]$ can take in-between values. This allows for the possibility that only a share of a representative household is eligible for pension payments in case of non-participation. The model can therefore capture different pension entry schemes. The first line in (12) reflects expected after-tax per-period labor income. During retirement, households can receive income-related pension benefits from different pension systems from the set $\mathcal{P}$. $P_t^{p,a}$ denotes the accumulated pension points according to pension system $p \in \mathcal{P}$ up to age a and time t. ς_t^a reflects how the stock of pension points translates into the yearly pension payment, which can depend on the retirement share (see next section). $\alpha_t^{p,a}$ is the share of household members subject to pension system p, with $\sum_p \alpha_t^{p,a} = 1$. In addition, the pension benefit can also consist of an exogenously given non-income-related component $P_t^{0,a}$ and survivors' pension $y_{surv,t}^a$ ($f_t^{w,a}$ being the share of widowed persons), which is computed as a fraction χ_t^w of the average survivors' pension income base $y_t^{w,a}$ (with $\phi_t^{w,a}$ governing the share of old-age pension income in that base). The gross income base is used to compute the accumulated points (see below). y_t^{cap} is the income threshold above which no pension rights are accumulated.[17] Note that pension claims are also collected during unemployment. In addition, pensions for orphans are taken into account ($f^{o,a}$ being the share of orphaned persons and χ^o the average orphan pension).[18] All pension systems are of the pay-as-you-go type, i.e., payments are financed out of current

[17]In the implementation, y_t^{cap} and therefore $y_t^{b,a}$ can differ by pension system type p, which is notationally omitted here.

[18]Note that we divide by the retirement share υ_t^a as the survivor's pension such that it cancels out υ_t^a in b^a, as – despite its name – the survivor's pension is actually received independently of whether or not a person is retired.

pension contributions from workers, firms, and the general budget. Note that retirees themselves do not pay pension contributions to the system anymore. The last term $\tau_t^{l,a}$ in (12) reflects lump-sum taxes (or transfers if negative) in terms of the consumption good, which do not alter any labor supply decision.

Retirement

The retirement share υ_t^a is increasing from 0 at $a = a^{R,early}$ to 1 at $a = a^{R,late}$. At age $a \in \left[a^{R,early}, a^{R,late}\right]$, the share of retired persons within a household increases by $d\upsilon_t^a = \upsilon_t^a - \upsilon_{t-1}^{a-1}$. Let $\hat{a}^{R,a}$ be the average retirement age of a household, measured at age a, averaging over all household members, and let $a_t^{R,a} = \hat{a}_t^{R,a}/\upsilon_t^a$ be the more relevant average retirement age measured at age a, conditional on being retired. They are computed as (omitting the time subscripts)

$$\hat{a}^{R,a} = \sum_{i=a^{R,early}}^{a} d\upsilon^i \cdot i, \quad \text{and} \quad a^{R,a} = \left[\sum_{i=a^{R,early}}^{a} d\upsilon^i \cdot i\right] / \upsilon^a. \tag{21}$$

In recursive terms, this can be rewritten as

$$\hat{a}^{R,a} = \hat{a}^{R,a-1} + d\upsilon^a \cdot a, \quad \text{and} \quad a^{R,a} = \left[a^{R,a-1} - a\right] \frac{\upsilon^{a-1}}{\upsilon^a} + a. \tag{22}$$

All related averaging works exactly the same way. Let $\varsigma^{GW,a}$ be the multiplicative adjustment factor for early or late retirement ('Gruber-Wise') if one retires at age a, and let $\varsigma^{CY,a}$ be the adjustment for having fewer contribution years than the target if one retires at age a (we omit time and pension type superscripts here). Then, the average adjustment factor ς^a is given as

$$\varsigma^a = \left[\varsigma^{a-1} - \varsigma^{GW,a}\varsigma^{CY,a}\right] \frac{\upsilon^{a-1}}{\upsilon^a} + \varsigma^{GW,a}\varsigma^{CY,a}. \tag{23}$$

Note that the derivative of $\varsigma^a \upsilon^a$ with respect to participation is

$$\frac{\partial(\varsigma^a \upsilon^a)}{\partial \delta^a} = \frac{\partial(\varsigma^a \upsilon^a)}{\partial \upsilon^a} \frac{\partial \upsilon^a}{\partial \delta^a} = -(1 - \phi^a)\varsigma^{GW,a}\varsigma^{CY,a}. \tag{24}$$

Earnings-related pensions

Pension payouts are directly related to households' earnings histories. This earnings-link already has important implications during the working life. In order to keep track of the earned pension rights, the stock variables $P_t^{p,a}$ are introduced.[19] The laws of motion for $P^{p,a}$ are given as

$$\mathcal{G}_t P_{t+1}^{p,a+1} = G_t^{p,a} \left[P_t^{p,a} + m_t^{p,a} \left(PBI_t^{p,a} + y_t^{0,p,a} \right) \right], \quad \forall p \in \mathcal{P}. \tag{25}$$

The factor $G_t^{p,a}$ determines the indexation of the earnings-related pension rights. $G_t^{p,a} = 1$ implies no real growth and can therefore be interpreted as inflation indexation. $G_t^{p,a} = \mathcal{G}_t$ implies that pension rights grow with inflation and labor productivity (which is about the same growth rate as for the wage rate). $m_t^{p,a}$ denotes the accumulation factor. $y_t^{0,p,a}$ is an exogenous pension base increment that is unrelated to the working history. This is used to incorporate the existence of a minimum pension and pension credits for child-rearing times. $PBI_t^{p,a}$ is the main pension base increment – the one depending on working history – which requires some deeper explanation. First, define

$$\mathbf{py}_t^{p,a} = \langle y_1, y_2, \ldots, y_{pn} \rangle \text{ with } y_1 \geq y_2 \geq \cdots \geq y_{pn}, \tag{26}$$

as the vector of totally ordered pensionable incomes of size pn. For example, if pension benefits are computed based on the 15 best income years, then $pn = 15$. Let $\Gamma_{pn}(\cdot)$ be a function that returns a vector of the pn largest elements, sorted from largest to smallest. Then, the law of motion for (detrended) pensionable incomes is

$$\mathcal{G}_t \mathbf{py}_{t+1}^{p,a+1} = G_t^{p,a} \Gamma_{pn}(\langle \mathbf{py}_t^{p,a}, y_t^{b,a} \rangle), \tag{27}$$

where $y_t^{b,a}$ is the gross income base. The pension base increment PBI_t^a (dropping the pension system index p for the moment) is then defined as

$$PBI_t^a = \max\{y_t^{b,a} - y_{pn}, 0\}, \tag{28}$$

where y_{pn} is the lowest current pensionable income, i.e., the last element of vector $\mathbf{py}_t^{p,a}$. This means that if the current gross income is larger than the smallest recorded pensionable income, the latter will be dropped in favor of the former; if not, the current income plays no role in computing pension benefits. The derivative

[19] We assume that migrants can credit their foreign pension claims, i.e., once they migrate, they have the same pension claims as domestic workers of the same age and skill. There is no reimbursement mechanism between the domestic and the foreign government, which could easily be implemented.

of PBI_t^a w.r.t. the gross income base $y_t^{b,a}$ is

$$\Upsilon_t^a \equiv \frac{\partial PBI_t^a}{\partial y_t^{b,a}} = \begin{cases} 1, & \text{if } \; y_t^{b,a} > y_{pn}, \\ 0, & \text{otherwise.} \end{cases} \tag{29}$$

The accumulation factor $m_t^{p,a}$ depends on the pension regime. In a system that grants 80% of the best 15 income years, this would, for example, be $m_t^{p,a} = 0.8/15$. The chosen specification also nests the case where all life-time income is taken into account for computing pension benefits ("Pensionskonto"), i.e., $pn = \overline{a} + 1$, which effectively implies $PBI_t^a = y_t^{b,a}$.

Types of labor

In production, different labor types are imperfectly substitutable. We assume that labor varies not only by skill but also by age. The age-dependence is approximated by using two types of labor: young (Y) and old (O).[20] We assume that one unit of labor from a household with age a has a young labor share of $\mu(a) \in [0, 1]$ and an old labor share of $1 - \mu(a)$. The share is given by a time-independent mapping: $\mu : \{\underline{a}, \ldots, \overline{a}\} \to [0, 1]$, with $\mu(\underline{a}) = 1$, $\mu(\overline{a}) = 0$, and $\mu(a_1) \leq \mu(a_2)$ if $a_1 \geq a_2$, i.e., weakly decreasing in age. We assume that the split between old and young is fixed for every unit of labor endowment, i.e., a household cannot decide to first supply only its old labor endowment and, only after its exhaustion, the young labor endowment. Hence, there is still only one labor supply decision at every margin based on the average return $w_t^{a,s}$. The average wage rate is simply $w_t^{a,s} = \mu(a) w_t^{Y,s} + (1 - \mu(a)) w_t^{O,s}$. Only this linear combination makes the household wage rate depend on single-year age, as the weights are functions of a. In total, the number of labor markets is $2 \cdot S$, which can be indexed by $j \in \mathcal{S} \times \{Y, O\}$. The formulation with only two fundamental age types of labor is, first, computationally much simpler than using individual single-age group labor markets, and second, well-suited to capture the idea that labor inputs of adjacent age groups are much closer substitutes than those of distant age groups.

[20] The model can easily be adapted to capture other types of complementarity, e.g., instead of imperfect substitution between young and old labor, one can implement imperfect substitution of domestic and foreign labor with minimal adjustments.

Household's decision problem

The problem of a representative household of type (a, s) – we drop the skill index s – can be written as follows:

$$V(A_t^a, \{P_t^{p,a}\}_{p\in\mathcal{P}}) = \max_{C_t^{U,a}, \ell_t^a, s_t^a, \delta_t^a} u(\tilde{C}_t^{U,a}, \Psi_t^a) + \beta\gamma_t^a \mathcal{G}_t^\rho V_{t+1}^{a+1}, \quad \text{s.t.} \tag{30}$$

$$(11), (25) \text{ and}$$

$$\tilde{C}_t^{U,a} = C_t^{U,a} - \kappa\bar{C}_{t-1}^{U,a-1}, \ \forall a > \underline{a} \quad \text{and} \quad \tilde{C}_t^{U,\underline{a}} = C_t^{U,\underline{a}}, \tag{31}$$

$$\Psi_t^a = \varphi^\delta(\delta_t^a) + \delta_t^a u_{t-1}^{a-1}\varphi^s(s_t^a) + \delta_t^a e_t^a \varphi^\ell(\ell_t^a) - \varphi^\upsilon(\upsilon_t^a). \tag{32}$$

Define $\sigma = 1/(1-\rho)$ as the intertemporal elasticity of substitution. Here, the household optimizes the allocation of composite consumption C over time, while the optimal composition of the consumption bundle can be solved at a later stage (see Section 2.10.1). We allow for the possibility of external habit persistence in consumption similar to Ratto et al. (2009) and based on Muellbauer (1988), where $\bar{C}_t^a$ is the average consumption of the respective household, which is taken as given. The parameter $\kappa > 0$ measures the degree of habit persistence. The model without habit formation is nested by setting $\kappa = 0$. $\varphi^\ell(\ell_t^a)$ is the disutility from working ℓ_t^a hours, $\varphi^s(s_t^a)$ is the disutility of searching with intensity s_t^a, and $\varphi^\delta(\delta_t^a)$ is the disutility from participating δ_t^a share of the period's time. All three are increasing and convex in their arguments.[21] In contrast, $\varphi^\upsilon(\upsilon_t^a)$ is the utility from being retired, which is increasing in remaining life expectancy and linear in the retirement share υ_t^a.

Define the change in remaining lifetime utility at time t due to a marginal increase in financial wealth and pension wealth as $\lambda_t^a \equiv \partial V_t^a/\partial A_t^a$ and $\eta_t^{p,a} \equiv \partial V_t^a/\partial P_t^{p,a}$,

[21]Note that in the implementation, the disutility functions are allowed to differ in age and time, i.e., $\varphi(\cdot)$ is short for $\varphi(\cdot; a, t)$.

respectively. The optimality and envelope conditions are

$$
\begin{aligned}
C_t^a: \quad u_{C_t^a} &= \gamma_t^a \beta \mathcal{G}_t^{\rho-1} \bar{R}_t^W \lambda_{t+1}^{a+1} p_t^C, && (33)\\
\ell_t^a: \quad -u_{\ell_t^a} &= \gamma_t^a \beta \mathcal{G}_t^{\rho-1} \bar{R}_t^W \lambda_{t+1}^{a+1} \cdot \frac{\partial y_t^a}{\partial \ell_t^a} + \\
&\quad \gamma_t^a \beta \mathcal{G}_t^{\rho-1} \cdot \frac{\partial y_t^{b,a}}{\partial \ell_t^a} \cdot \sum_{p \in \mathcal{P}} G_t^{p,a} \eta_{t+1}^{p,a+1} m_t^{p,a} \Upsilon_t^{p,a}, && (34)\\
s_t^a: \quad -u_{s_t^a} &= \gamma_t^a \beta \mathcal{G}_t^{\rho-1} \bar{R}_t^W \lambda_{t+1}^{a+1} \cdot \frac{\partial y_t^a}{\partial s_t^a} + \\
&\quad \gamma_t^a \beta \mathcal{G}_t^{\rho-1} \cdot \frac{\partial y_t^{b,a}}{\partial s_t^a} \cdot \sum_{p \in \mathcal{P}} G_t^{p,a} \eta_{t+1}^{p,a+1} m_t^{p,a} \Upsilon_t^{p,a}, && (35)\\
\delta_t^a: \quad -u_{\delta_t^a} &= \gamma_t^a \beta \mathcal{G}_t^{\rho-1} \bar{R}_t^W \lambda_{t+1}^{a+1} \cdot \frac{\partial y_t^a}{\partial \delta_t^a} + \\
&\quad \gamma_t^a \beta \mathcal{G}_t^{\rho-1} \cdot \frac{\partial y_t^{b,a}}{\partial \delta_t^a} \cdot \sum_{p \in \mathcal{P}} G_t^{p,a} \eta_{t+1}^{p,a+1} m_t^{p,a} \Upsilon_t^{p,a}, && (36)\\
A_t^a: \quad \lambda_t^a &= \gamma_t^a \beta \mathcal{G}_t^{\rho-1} \bar{R}_t^W \lambda_{t+1}^{a+1}, && (37)\\
P_t^{p,a}: \quad \eta_t^{p,a} &= \gamma_t^a \beta \mathcal{G}_t^{\rho-1} \times \Bigg[G_t^{p,a} \eta_{t+1}^{p,a+1} + \\
&\quad \bar{R}_t^W \lambda_{t+1}^{a+1} \upsilon_t^a (1-\tau_t^{P,a}) \alpha_t^{p,a} \varsigma_t^{p,a} \Bigg], \quad \forall p \in \mathcal{P}, && (38)
\end{aligned}
$$

with marginal changes in income y_t^a and the gross income base, $y_t^{b,a}$ of

$$
\begin{aligned}
\frac{\partial y_t^a}{\partial \ell_t^a} &= \delta_t^a e_t^a (1-\tilde{\tau}_t^{W,a}) w_t^a \theta_t^a,\\
\frac{\partial y_t^a}{\partial s_t^a}, &= \delta_t^a u_{t-1}^{a-1} q(\theta_t) \left[(1-\tau_t^{W,a}) w_t^a \theta_t^a \ell_t^a - b_t^{u,a} \right],\\
\frac{\partial y_t^a}{\partial \delta_t^a} &= e_t^a (1-\tau_t^{W,a}) w_t^a \theta_t^a \ell_t^a + u_t^a b_t^{u,a} - \tilde{b}_t^a,\\
\frac{\partial y_t^{b,a}}{\partial \ell_t^a} &= \delta_t^a e_t^a w_t^a \theta_t^a,\\
\frac{\partial y_t^{b,a}}{\partial s_t^a} &= \delta_t^a u_{t-1}^{a-1} q(\theta_t) \left[w_t^a \theta_t^a \ell_t^a - b_t^{p,a} \right],\\
\frac{\partial y_t^{b,a}}{\partial \delta_t^a} &= e_t^a w_t^a \theta_t^a \ell_t^a + u_t^a b_t^{p,a},
\end{aligned}
$$

where

$$\tilde{b}_t^a \equiv \phi_t^a b_t^{n,a} - b_t^{k,a} + (1-\phi_t^a)(1-\tau_t^{P,a}) \sum_{p \in \mathcal{P}} \alpha_t^{p,a} P_t^{p,a} \varsigma_t^{GW,p,a} \varsigma_t^{CY,p,a}, \tag{39}$$

is the marginal gain from non-participation, in contrast to the average gain b_t^a.[22] Note that for the intensive margin, the marginal tax rate $\tilde{\tau}_t^{W,a}$ is applied (see Section 3.4).[23] Instantaneous utility functions $u(\cdot)$ can be of two types: with or without income effects of labor supply. The assumed functional forms can be found in Section 3.1. The important thing is that, in both cases, the ratios of marginal utilities can be reduced to the following:

$$-\frac{u_{\ell_t^a}}{u_{C_t^a}} = \delta_t^a e_t^a \varphi^{\ell\prime}(\ell_t^a) \Xi_t^a, \tag{40}$$

$$-\frac{u_{s_t^a}}{u_{C_t^a}} = \left[\delta_t^a u_{t-1}^{a-1} \varphi^{s\prime}(s_t^a) + \delta_t^a u_{t-1}^{a-1} q(\theta_t) \varphi^{\ell}(\ell_t^a)\right] \Xi_t^a, \tag{41}$$

$$-\frac{u_{\delta_t^a}}{u_{C_t^a}} = \left[\varphi^{\delta\prime}(\delta_t^a) + u_{t-1}^{a-1} \varphi^{s}(s_t^a) + e_t^a \varphi^{\ell}(\ell_t^a) + (1-\phi_t^a) \varphi^{\upsilon\prime}(\upsilon_t^a)\right] \Xi_t^a, \tag{42}$$

with $\Xi_t^a = \tilde{C}_t^a$ in the case of income effects, and $\Xi_t^a = 1$ in the case without income effects. It is convenient to express the shadow price of pension points in relative terms by dividing (38) by (37) and defining $\tilde{\eta}_t^{p,a} = \eta_t^{p,a}/\lambda_t^a, \ \forall p \in \mathcal{P}$:

$$\tilde{\eta}_t^{p,a} = \upsilon_t^a (1-\tau_t^{P,a}) \alpha_t^{p,a} \varsigma_t^{p,a} + \frac{G^{p,a}}{\bar{R}_t^W} \tilde{\eta}_{t+1}^{p,a+1}. \tag{43}$$

Labor supply at the intensive margin

Divide (34) by (33) and insert (40) to arrive at the following labor supply condition for hours:

$$p_t^C \varphi^{\ell\prime}(\ell_t^a) \Xi_t^a = (1-\tau_t^{W,a}) w_t^a \theta_t^a + \Theta_t^a w_t^a \theta_t^a, \tag{44}$$

[22]The flat pension part $P^{a,0}$ plays no role in $\tilde{b}_t^a$. Further, the marginal adjustment factor $\varsigma_t^{GW,p,a} \varsigma_t^{CY,p,a}$ matters here instead of the average one $\varsigma_t^{p,a}$.

[23]Observe the difference between 'marginal' in terms of participation and hours. The first refers to a marginal change in the number of household members that participate, while the latter refers to a marginal change in hours supplied, conditional on employment. As labor taxation occurs at the individual household member level, the marginal tax rate enters the first order condition for hours, while the average tax rate pops up in the first order condition for participation and search.

where Θ_t^a summarizes the tax-benefit link of the pension system:

$$\Theta_t^a \equiv \frac{\sum_p m_t^{p,a} \Upsilon_t^{p,a} G_t^{p,a} \tilde{\eta}_{t+1}^{p,a+1}}{\bar{R}_t^W}. \tag{45}$$

This can be rewritten as

$$\varphi^{\ell\prime}(\ell_t^a)\Xi_t^a = (1 - \hat{\tau}_t^{W,a}) w_t^a \theta_t^a / \tilde{p}_t^C, \tag{46}$$

with an effective tax rate of

$$\hat{\tau}_t^{\ell,a} = \frac{\tau_t^C + \tilde{\tau}_t^{W,a} - \Theta_t^a}{1 + \tau_t^C}. \tag{47}$$

Clearly, the higher $m_t^{p,a}$, the stronger the tax-benefit link; the lower the effective tax rate, i.e., with a strong earnings link, the contributions to the pension system are less perceived as taxes.

Labor supply at the search effort margin

Proceed as before to express the first-order condition for search effort as

$$\begin{aligned} &\left[\varphi^{s\prime}(s_t^a) + q(\theta_t)\varphi^{\ell}(\ell_t^a)\right] \Xi_t^a = \\ &\qquad q(\theta_t)\left[(1 - \tau_t^{W,a}) w_t^a \theta_t^a \ell_t^a - b_t^{u,a} + \Theta_t^a \left(w_t^a \theta_t^a \ell_t^a - b_t^{p,a}\right)\right] / p_t^C. \end{aligned} \tag{48}$$

This can be rewritten as

$$\varphi^{s\prime}(s_t^a)\Xi_t^a = q(\theta_t)(1 - \hat{\tau}_t^{s,a}) w_t^a \ell_t^a \theta_t^a / \tilde{p}_t^C - q(\theta_t)\varphi^{\ell}(\ell_t^a)\Xi_t^a, \tag{49}$$

with an effective tax on search effort equal to

$$\hat{\tau}_t^{s,a} = \frac{\tau_t^C + \tau_t^{W,a} + b_t^{u,a}/(w_t^a \ell_t^a \theta_t^a) - \Theta_t^a \left[1 - b_t^{p,a}/(w_t^a \ell_t^a \theta_t^a)\right]}{1 + \tau_t^C}, \tag{50}$$

or, after using (18) and (19), we have

$$\hat{\tau}_t^{s,a} = \frac{\tau_t^C + \tau_t^{W,a} + (1 - \tau^{W,a})\phi_t^{u,a} - \Theta_t^a \left[1 - \phi_t^{p,a}\right]}{1 + \tau_t^C}. \tag{51}$$

Labor supply at the extensive margin

The participation decision occurs prior to the hours and search effort decision. The representative household chooses a share δ_t^a of household members that participate in the labor market during a period. Divide (36) by (33) and insert (42) to arrive at the following labor supply condition for participation:

$$\left[\varphi^{\delta\prime}(\delta_t^a) + u_{t-1}^{a-1}\varphi^s(s_t^a) + e_t^a\varphi^\ell(\ell_t^a) + (1-\phi_t^a)\varphi^{\upsilon\prime}(\upsilon_t^a)\right]\Xi_t^a = \left[e_t^a(1-\tau_t^{W,a})w_t^a\ell_t^a\theta_t^a + u_t^a b_t^{u,a} - \tilde{b}_t^a + \Theta_t^a(e_t^a w_t^a \ell_t^a\theta_t^a + u_t^a b_t^{p,a})\right]/p_t^C. \tag{52}$$

This can be rewritten as

$$\varphi^{\delta\prime}(\delta_t^a)\Xi_t^a = (1-\hat{\tau}_t^{\delta,a})e_t^a w_t^a\ell_t^a\theta_t^a/\tilde{p}_t^C - \left[e_t^a\varphi^\ell(\ell_t^a) + u_{t-1}^{a-1}\varphi^s(s_t^a) + (1-\phi_t^a)\varphi^{\upsilon\prime}(\upsilon_t^a)\right]\Xi_t^a. \tag{53}$$

with a participation tax equal to

$$\hat{\tau}_t^{\delta,a} = \frac{\tau_t^C + \tau_t^{W,a} + (\tilde{b}_t^a - u_t^a b_t^{u,a})/(e_t^a w_t^a\ell_t^a\theta_t^a) - \Theta_t^a\left[1 + u_t^a b_t^{p,a}/(e_t^a w_t^a\ell_t^a\theta_t^a)\right]}{1+\tau_t^C}, \tag{54}$$

or after using (18) and (19),

$$\hat{\tau}_t^{\delta,a} = \frac{\tau_t^C + \tau_t^{W,a} + \tilde{b}_t^a/(e_t^a w_t^a\ell_t^a\theta_t^a) - u_t^a/e_t^a(1-\tau_t^{W,a})\phi_t^{u,a} - \Theta_t^a\left[1 + u_t^a/e_t^a\phi_t^{p,a}\right]}{1+\tau_t^C}. \tag{55}$$

Consumption

Combine (33) and (37) to get $u_{C_t^a} = \lambda_t^a p_t^C$. Use this expression again in (37) to derive the Euler equations (see Section 3.1 for functional forms).

Euler equation (income effects specification):

$$\mathcal{G}_t\tilde{C}_{t+1}^{U,a+1} = \left[\frac{p_t^C}{p_{t+1}^C}\gamma_t^a\hat{\beta}_t^a\bar{R}_t^W\right]^\sigma \left[e^{\Psi_{t+1}^{a+1}-\Psi_t^a}\right]^{1-\sigma}\tilde{C}_t^{U,a}. \tag{56}$$

If $\sigma = 1$ (log-specification), then the disutility of labor has no influence on the shape of the consumption profile. In the numerical implementation of the income effects specification, labor supply and consumption have to be solved simultaneously using policy function iteration. $\hat{\beta}_t^a \equiv \beta\frac{\omega_{t+1}^{a+1}}{\omega_t^a}$ is the effective discount factor, taking changes in household size weights into account (see Section 3.1).

Euler equation (no income effects specification):

$$\mathcal{G}_t \left[\tilde{C}^{U,a+1}_{t+1} - \Psi^{a+1}_{t+1}\right] = \left[\frac{p^C_t}{p^C_{t+1}} \gamma^a_t \hat{\beta}^a_t \bar{R}^W_t\right]^\sigma \left[\tilde{C}^{U,a}_t - \Psi^a_t\right]. \tag{57}$$

Lemma 2.1. *Consumption function (no income effect specification). The choice of the current consumption-leisure bundle can be analytically expressed as a function of current assets A^a_t and discounted human wealth H^a_t.*

$$C^{U,a}_t = \left(\Omega^a_t p^C_t\right)^{-1} \left(A^a_t + H^a_t - p^C_t \Gamma^a_t \kappa C^{U,a-1}_{t-1}\right) + \Psi^a_t + C^{U,a-1}_{t-1}, \tag{58}$$

with Γ^a_t and H^a_t as defined below.

Proof is provided in the appendix. The recursive solution block is given by (2.1), and

$$\Gamma^a_t = 1 + \kappa \left[\frac{p^C_{t+1}}{p^C_t} \frac{\mathcal{G}_t}{\bar{R}^W_t}\right] \Gamma^{a+1}_{t+1}, \quad \text{with} \quad \Gamma^{\bar{a}}_t = 1, \tag{59}$$

$$\Omega^a_t = \Gamma^a_t + \left(\gamma^a_t \hat{\beta}^a_t\right)^\sigma \left(\bar{R}^W_t \frac{p^C_t}{p^C_{t+1}}\right)^{\sigma-1} \Omega^{a+1}_{t+1}, \tag{60}$$

$$H^a_t = \bar{y}^a_t - \Psi^a_t \Gamma^a_t p^C_t + \frac{\mathcal{G}_t H^{a+1}_{t+1}}{\bar{R}^W_t}. \tag{61}$$

Note that for the numerical implementation, it is convenient to rewrite equations (59) and (60) such that consumption prices are part of the foresight variable.

$$\Lambda^a_t = \Delta^a_t (p^C_t)^{-\sigma} + \left(\gamma^a_t \hat{\beta}^a_t\right)^\sigma \left(\bar{R}^W_t\right)^{\sigma-1} \Lambda^{a+1}_{t+1}, \quad \text{where} \quad \Lambda^a_t = \Omega^a_t (p^C_t)^{1-\sigma} \tag{62}$$

$$\Delta^a_t = p^C_t + \kappa \mathcal{G}_t / \bar{R}^W_t \Delta^{a+1}_{t+1}, \quad \text{where} \quad \Delta^a_t = \Gamma^a_t p^C_t. \tag{63}$$

2.6.2 'Rule of Thumb' Households

'Rule of thumb' (or constrained) households do not save or borrow and therefore spend all of their disposable income in the same period. This can be motivated by liquidity constraints or by assuming cognitive restrictions which prevent these households from performing a full intertemporal optimization and consumption smoothing. Another motivation for incorporating constrained households is to replicate realistic consumption behavior. If we had only unconstrained households, households' consumption behavior would follow the permanent income hypothesis, e.g., windfall gains are perfectly smoothed over the whole remaining life-cycle or future income increases have similarly strong effects on today's consumption as on consumption in the period when the income increase actually occurs. The model set-up follows Campbell and Mankiw (1989), Mankiw (2000), Galí et al. (2007) and others. Card et al. (2007) shows that empirical consumption behavior is somewhere between predictions of the permanent income hypothesis and hand-to-mouth consumption. The parameter π^s can then used to target the empirical consumption behavior.

The consumption function for constrained households (dropping skill indices) is simply

$$C_t^{C,a} = \bar{y}_t^{C,a}/p_t^C, \quad \bar{y}^{C,a} = y_t^a - Z_t^{C,a}. \tag{64}$$

With respect to labor supply, the model features two options. First, constrained households just follow the unconstrained households, which considerably reduces computational complexity (*default option*); or second, they do their own full optimization w.r.t. $\ell^{C,a}$ and $\delta^{C,a}$, in which case pension points, their related shadow prices, etc., will differ from their unconstrained counterparts (and therefore have to be superscripted accordingly). The optimization problem is the same except for (64) replacing (11) and adjusting superscripts accordingly. The optimality conditions (omitting the C-superscript for all endogenous variables for easier readability) are:

$$\begin{aligned}
\ell_t^a : \ -u_{\ell_t^a} &= u_{C_t^a}/p_t^C \cdot \frac{\partial y_t^a}{\partial \ell_t^a} + \gamma_t^a \beta \mathcal{G}_t^{\rho-1} \cdot \frac{\partial y_t^{b,a}}{\partial \ell_t^a} \cdot \sum_{p \in \mathcal{P}} G_t^{p,a} \eta_{t+1}^{p,a+1} m_t^{p,a} \Upsilon_t^{p,a}, \\
s_t^a : \ -u_{s_t^a} &= u_{C_t^a}/p_t^C \cdot \frac{\partial y_t^a}{\partial s_t^a} + \gamma_t^a \beta \mathcal{G}_t^{\rho-1} \cdot \frac{\partial y_t^{b,a}}{\partial s_t^a} \cdot \sum_{p \in \mathcal{P}} G_t^{p,a} \eta_{t+1}^{p,a+1} m_t^{p,a} \Upsilon_t^{p,a}, \\
\delta_t^a : \ -u_{\delta_t^a} &= u_{C_t^a}/p_t^C \cdot \frac{\partial y_t^a}{\partial \delta_t^a} + \gamma_t^a \beta \mathcal{G}_t^{\rho-1} \cdot \frac{\partial y_t^{b,a}}{\partial \delta_t^a} \cdot \sum_{p \in \mathcal{P}} G_t^{p,a} \eta_{t+1}^{p,a+1} m_t^{p,a} \Upsilon_t^{p,a}.
\end{aligned}$$

Rearranging and inserting (40) and (42) gives the same labor supply conditions (46) and (53) as for the unconstrained households, with a different expression for the pension link Θ:

$$\Theta_t^a \equiv p_t^C \beta \gamma_t^a \mathcal{G}_t^{\rho-1} \frac{\sum_p m_t^{p,a} \Upsilon_t^a G_t^{p,a} \eta_{t+1}^{p,a+1}}{u_{C_t^a}}. \tag{65}$$

2.7 Accidental Bequests

We assume the following timing: at the beginning of a period, households receive accidental bequests at the same time as other income flows and start consuming. At the end of the period, some households die and leave their savings S_t^a,

$$S_t^a = A_t^a + \bar{y}_t^{U,a} - p_t^C C_t^{U,a}. \quad (66)$$

This implies that at the end of every period, the following total assets are collected: $\sum_{s=1}^{S} \sum_{a=\underline{a}}^{\bar{a}} (1 - \gamma_t^{a,s}) S_t^{a,s} N_t^{U,a,s}$. Hence, the condition that equates accidental assets received and given must hold.

$$\sum_{s=1}^{S} \sum_{a=\underline{a}}^{\bar{a}} ab_t^{a,s} N_t^{U,a,s} = \sum_{s=1}^{S} \sum_{a=\underline{a}}^{\bar{a}} (1 - \gamma_t^{a,s}) S_t^{a,s} N_t^{U,a,s}. \quad (67)$$

More specifically, one can assume a distribution rule of the following form:

$$ab_t^{a,s} = \xi_t^{a,s} \cdot \frac{\sum_{s=1}^{S} \sum_{a=\underline{a}}^{\bar{a}} (1 - \gamma_t^{a,s}) S_t^{a,s} N_t^{U,a,s}}{N_t^{U,a,s}}, \qquad \sum_{a=\underline{a}}^{\bar{a}} \sum_{s=1}^{S} \xi_t^{a,s} = 1, \quad (68)$$

where $\xi_t^{a,s}$ denotes some exogenous weights.

2.8 Aggregation

Labor force by skill and age labor type is $LF_t^{Y,s} = \sum_{a \in \mathcal{A}} \delta_t^{a,s} N_t^{a,s} \mu(a)$ and $LF_t^{O,s} = \sum_{a \in \mathcal{A}} \delta_t^{a,s} N_t^{a,s} (1 - \mu(a))$. Aggregate effective labor supply by skill and labor type is given by $L_t^{S,Y,s} = \sum_{a \in \mathcal{A}} \delta_t^{a,s} e_t^{Y,a,s} \ell_t^{a,s} \theta_t^{a,s} N_t^{a,s} \mu(a)$ and $L_t^{S,O,s} = \sum_{a \in \mathcal{A}} \delta_t^{a,s} e_t^{O,a,s} \ell_t^{a,s} \theta_t^{a,s} N_t^{a,s} (1- \mu(a))$. These expressions have to equal labor demands $L_t^{D,Y,s}$ and $L_t^{D,O,s}, \forall s \in \mathcal{S}$ in equilibrium. The employment and the wage rate, averaged over labor types, are simply $e_t^{a,s} = \mu(a) e_t^{Y,a,s} + (1 - \mu(a)) e_t^{O,a,s}$ and $w_t^{a,s} = \mu(a) w_t^{Y,a,s} + (1 - \mu(a)) w_t^{O,a,s}$, respectively. Aggregate total effective labor supply is $L_t^S = \sum_{i \in \mathcal{A} \times \mathcal{S}} \delta_t^i e_t^i \ell_t^i \theta_t^i N_t^i$. The wage sum is given as $L_t^{Sum} = \sum_{i \in \mathcal{A} \times \mathcal{S}} w_t^i \delta_t^i e_t^i \ell_t^i \theta_t^i N_t^i$. It is handy to define the average labor and contribution tax rates over all ages paid by the workers as $\tau_t^W = \frac{\sum_{i \in \mathcal{A} \times \mathcal{S}} \tau_t^{W,i} w_t^i \delta_t^i e_t^i \ell_t^i \theta_t^i N_t^i}{L_t^{Sum}}$, and the average wage rate as $w_t = \frac{\sum_{i \in \mathcal{A} \times \mathcal{S}} w_t^i \delta_t^i e_t^i \ell_t^i \theta_t^i N_t^i}{L_t^S}$. This way, one can conveniently write aggregate after-tax and contribution labor income as $w_t(1 - \tau_t^W) L_t^S$. Similarly, $\tau^{W,c}$ is used as notation for the average pension contribution rate. All other variables are aggregated by adding all (adult) age groups and skill classes, i.e., for some variable X, we define

$$X_t = \sum_{s=1}^{S} \sum_{a=\underline{a}}^{\bar{a}} X_t^{a,s} N_t^{a,s} = \sum_{i \in \mathcal{A}\times\mathcal{S}} X_t^i N_t^i. \tag{69}$$

For consumption and population size, we also have to aggregate over unconstrained and constrained households. To compute aggregate end-of-period savings, we have to take into account that migrants of age a and skill s arrive with average assets of the representative household of the same age and skill. Recall that only the share of migrants that is unconstrained brings along assets. Aggregate end-of-period savings are given as

$$S_t = \left[A_t + A_t^{Mig} - Z_t + y_t - p_t^C C_t\right] = \sum_{i \in \mathcal{A}\times\mathcal{S}} \gamma_t^i S_t^i N_t^{U,i} + A_t^{Mig}, \tag{70}$$

$$A_t^{Mig} \equiv \mathcal{N}_t \sum_{s=1}^{S} \sum_{a=\underline{a}}^{\bar{a}} Mig_{t+1}^{U,a+1,s} \cdot S_t^{a,s} \qquad Z_t = \sum_{s=1}^{S} \sum_{a=\underline{a}}^{\bar{a}} \sum_{k\in\{U,C\}} Z_t^{k,a,s} N_t^{k,a,s},$$

where A_t^{Mig} is the inflowing (net) assets of (net) migration at the end of period t, and $S_t^{a,s}$ is the end-of-period assets (i.e., savings in period t), as given in (66). Z_t is the aggregate foreign financial transfers made.

Lemma 2.2. ***Aggregate assets*** *evolve according to the following equation*

$$\hat{\mathcal{G}}_t A_{t+1} = \bar{R}_t^W \cdot S_t, \quad S_t = \left[A_t + A_t^{Mig} - Z_t + y_t - p_t^C C_t\right], \tag{71}$$

where $\hat{\mathcal{G}}_t = \mathcal{G}_t \mathcal{N}_t$ *and* $\hat{g}_t = \hat{\mathcal{G}}_t - 1$.

Proof is provided in the appendix. An alternative way of writing the law of motion of aggregate assets is

$$\hat{\mathcal{G}}_t A_{t+1} = R_t^W \cdot \left[S_t - T_t^R\right] = R_t^W \cdot \hat{A}_t, \tag{72}$$

where $\hat{A}$ is the end of period assets after interest tax payments T^R (in contrast to S), and R^W is the before-tax interest factor. By inserting (67), aggregate end of period assets can alternatively be written as

$$S_t = \sum_{s=1}^{S} \sum_{a=\underline{a}}^{\bar{a}} \gamma_t^{a,s} S_t^{a,s} N_t^{U,a,s} + A_t^{Mig} = \sum_{i \in \mathcal{A}\times\mathcal{S}} \left[S_t^i - ab_t^i\right] \cdot N_t^{U,i} + A_t^{Mig}. \tag{73}$$

2.9 Portfolio Choice[24]

Households allocate their savings into four different asset types: domestic and foreign government bonds, and domestic and foreign firm assets. The asset types are imperfect substitutes. This breaks the equality of net-of-tax interest rates by non-arbitrage. We assume that households have the same portfolio preferences, and therefore, the allocation problem can be solved at the aggregate level instead of the individual household level. We use a two-level CES specification for household preferences similar to Keuschnigg and Dietz (2007). At the upper level, end-of-period assets (after interest taxation) $\hat{A}$ are split into demand for government bonds $\hat{A}^G$ and firm assets $\hat{A}^V$. At the lower level, both are again split into home and foreign demand, i.e., $\hat{A}^V$ into $\hat{A}^{V,h}$ and $\hat{A}^{V,m}$, and $\hat{A}^G$ into $\hat{A}^{G,h}$ and $\hat{A}^{G,m}$. These are the demands for home and foreign assets by domestic households and should not be confused with the demand for domestic assets from abroad.

$$\hat{A} = \hat{A}^V + \hat{A}^G, \quad \hat{A}^V = \hat{A}^{V,h} + \hat{A}^{V,m}, \quad \hat{A}^G = \hat{A}^{G,h} + \hat{A}^{G,m} \quad \text{or} \tag{74}$$

$$\hat{A} = \hat{A}^{V,h} + \hat{A}^{V,m} + \hat{A}^{G,h} + \hat{A}^{G,m}. \tag{75}$$

Let $\bar{R}^i$, with $i \in \{V, G, (V,h), (V,m), (G,h), (G,m)\}$, denote the corresponding after-tax interest factors, while the 'without bar' notation refers to before-tax rates and factors. We assume taxation of the nominal interest rate. In general, this means that starting from the before-tax real rate $r_t = \frac{i_t - g^€}{\mathcal{G}^€}$, we get the following after-tax real rate $\bar{r}_t = \frac{i_t(1-\tau_t^R) - g^€}{\mathcal{G}^€}$.[25] In particular, we assume

$$\bar{r}^{V,h} = \frac{i^{V,h}(1-\tau^{R,V,h}) - g^€}{\mathcal{G}^€}, \qquad \bar{R}^{V,h} = 1 + \bar{r}^{V,h}, \qquad R^{V,h} = 1 + r^{V,h}, \tag{76}$$

$$\bar{r}^{V,m} = \frac{i^{V,m}(1-\tau^{R,V,m}) - g^€}{\mathcal{G}^€}, \qquad \bar{R}^{V,m} = 1 + \bar{r}^{V,m}, \qquad R^{V,h} = 1 + r^{V,m}, \tag{77}$$

$$\bar{r}^{G,h} = \frac{i^{G,h}(1-\tau^{R,G,h}) - g^€}{\mathcal{G}^€}, \qquad \bar{R}^{G,h} = 1 + \bar{r}^{G,h}, \qquad R^{G,h} = 1 + r^{G,h}, \tag{78}$$

$$\bar{r}^{G,m} = \frac{i^{G,m}(1-\tau^{R,G,m}) - g^€}{\mathcal{G}^€}, \qquad \bar{R}^{G,m} = 1 + \bar{r}^{G,m}, \qquad R^{G,m} = 1 + r^{G,m}. \tag{79}$$

At the **upper level**, asset allocation is based on utility maximization from investing

[24] We drop the time index throughout this section for convenience because of the static nature of the problems.

[25] Note the difference compared to taxing real returns, in which case we would have $\bar{r}_t = \frac{(i_t - g^€)(1-\tau_t^R)}{\mathcal{G}^€} = (1-\tau_t^R) r_t$. In contrast, with taxation of nominal returns, we have $\bar{r}_t = (1-\tau_t^R) r_t - \tau_t^R g^€ / \mathcal{G}^€$.

one unit of assets[26] of the following form, i.e.,

$$W = \max_{a^V, a^G} \left[(\xi^A)^{1-\varepsilon^A} \left(\bar{R}^V a^V \right)^{\varepsilon^A} + (1-\xi^A)^{1-\varepsilon^A} \left(\bar{R}^G a^G \right)^{\varepsilon^A} \right]^{1/\varepsilon^A} \tag{80}$$

subject to $a^V + a^G = 1$. Let $\sigma^A = 1/(1-\varepsilon^A)$ denote the elasticity of substitution. The solutions for investment shares are

$$a^V = \xi^A \cdot \left[\tilde{R}/\bar{R}^V \right]^{1-\sigma^A}, \quad a^G = (1-\xi^A) \cdot \left[\tilde{R}/\bar{R}^G \right]^{1-\sigma^A}, \tag{81}$$

with a composite index $\tilde{R}$ given as

$$\tilde{R} = \left[\xi^A \left(\bar{R}^V \right)^{\sigma^A - 1} + (1-\xi^A) \left(\bar{R}^G \right)^{\sigma^A - 1} \right]^{1/(\sigma^A - 1)}. \tag{82}$$

At the **bottom level**, the allocation problem works analogously for $i = V, G$.

$$W^i = \max_{a^{i,h}, a^{i,m}} \left[(\xi^{A^i})^{1-\varepsilon^{A^i}} \left(\bar{R}^{i,h} a^{i,h} \right)^{\varepsilon^{A^i}} + (1-\xi^{A^i})^{1-\varepsilon^{A^i}} \left(\bar{R}^{i,m} a^{i,m} \right)^{\varepsilon^{A^i}} \right]^{1/\varepsilon^{A^i}}.$$

subject to $a^{i,h} + a^{i,m} = 1$ and an elasticity of substitution of $\sigma^{A^i} = 1/(1-\varepsilon^{A^i})$. The solutions are given as

$$a^{i,h} = \xi^{A^i} \cdot \left[\tilde{R}^i/\bar{R}^{i,h} \right]^{1-\sigma^{A^i}}, \quad a^{i,m} = (1-\xi^{A^i}) \cdot \left[\tilde{R}^i/\bar{R}^{i,m} \right]^{1-\sigma^{A^i}}, \tag{83}$$

with composite indices $\tilde{R}^i$ given as

$$\tilde{R}^i = \left[\xi^{A^i} \left(\bar{R}^{i,h} \right)^{\sigma^{A^i} - 1} + (1-\xi^{A^i}) \left(\bar{R}^{i,m} \right)^{\sigma^{A^i} - 1} \right]^{1/(\sigma^{A^i} - 1)}. \tag{84}$$

We therefore have the following asset allocation:

$$\hat{A}^V = a^V \cdot \hat{A}, \quad \hat{A}^G = a^G \cdot \hat{A}. \tag{85}$$

$$\hat{A}^{V,h} = a^{V,h} \hat{A}^V, \quad \hat{A}^{V,m} = a^{V,m} \hat{A}^V, \quad \hat{A}^{G,h} = a^{G,h} \hat{A}^G, \quad \hat{A}^{G,m} = a^{G,m} \hat{A}^G. \tag{86}$$

Importantly, other economic decisions of the households, such as saving, are based

[26] Observe that this is equivalent to the problem of investing some A because of the linear homogeneity of the utility function.

on the average portfolio return $\bar{R}^W \neq \tilde{R}$.

$$\bar{R}^W = \left[\bar{R}^{V,h} a^V a^{V,h} + \bar{R}^{V,m} a^V a^{V,m} + \bar{R}^{G,h} a^G a^{G,h} + \bar{R}^{G,m} a^G a^{G,m}\right], \tag{87}$$
$$\bar{r}^W = \left[\bar{r}^{V,h} a^V a^{V,h} + \bar{r}^{V,m} a^V a^{V,m} + \bar{r}^{G,h} a^G a^{G,h} + \bar{r}^{G,m} a^G a^{G,m}\right]. \tag{88}$$

One can further define the average return before tax, i.e.,

$$R^W = \left[R^{V,h} a^V a^{V,h} + R^{V,m} a^V a^{V,m} + R^{G,h} a^G a^{G,h} + R^{G,m} a^G a^{G,m}\right], \tag{89}$$
$$r^W = \left[r^{V,h} a^V a^{V,h} + r^{V,m} a^V a^{V,m} + r^{G,h} a^G a^{G,h} + r^{G,m} a^G a^{G,m}\right], \tag{90}$$
$$i^W = \left[i^{V,h} a^V a^{V,h} + i^{V,m} a^V a^{V,m} + i^{G,h} a^G a^{G,h} + i^{G,m} a^G a^{G,m}\right]. \tag{91}$$

The average interest tax rate on interest is, therefore,

$$\tau_t^R = \left[\sum_{i \in \{V,G\}} \sum_{j \in \{h,m\}} \tau_t^{R,i,j} \cdot i_t^{i,j} \cdot a_t^i a_t^{i,j}\right] / i_t^W. \tag{92}$$

2.10 Final Demands[27]

Demand for final goods can stem from different sources: private consumption, private investment, public consumption, public investment, and foreign sources. We assume the same class of subutility/production functions of CES-form but allow for different parameterizations depending on the source of demand. In particular, we can set different import shares and different elasticities for substitution between domestic and imported bundles. We assume the same subutility parameters, independent of age, skill, and whether a household is constrained or not. Using linear homogeneous CES-aggregators, this implies that we can directly disentangle aggregate demand for the composite consumption good C (instead of disentangling individual demands and aggregating afterward).

2.10.1 Consumption

We assume preferences concerning the composite consumption good of the following CES form:

$$C = \left[(\xi^C)^{1-\varepsilon^C} \left(C^m\right)^{\varepsilon^C} + (1-\xi^C)^{1-\varepsilon^C} \left(C^h\right)^{\varepsilon^C}\right]^{1/\varepsilon^C}. \tag{93}$$

[27]We drop the time index throughout this section for convenience because of the static nature of the problems.

C^m and C^h are imported and domestically produced quantities. Next, we compute compensated unit demands c^m and c^h and the unit expenditure function by solving

$$p^C = \min_{c^m, c^h} \left\{ p^{C,m} c^m + p^{C,h} c^h \right\} \quad \text{s.t.} \quad C(c^m, c^h) = 1. \tag{94}$$

As a solution, we get the usual unit expenditure function p^C and unit demand functions for the CES form.

$$p^C = \left[\xi^C \left(p^{C,m} \right)^{1-\lambda^C} + (1 - \xi^C) \left(p^{C,h} \right)^{1-\lambda^C} \right]^{\frac{1}{1-\lambda^C}}, \tag{95}$$

$$c^m = \xi^C \left[p^C / p^{C,m} \right]^{\lambda^C}, \quad c^h = (1 - \xi^C) \left[p^C / p^{C,h} \right]^{\lambda^C}, \tag{96}$$

where $\lambda^C = 1/(1 - \varepsilon^C)$. The solution for splitting composite consumption is then simply $C^h = c^h \cdot C$ and $C^m = c^m \cdot C$. Input prices include product taxes, i.e., $p^{C,h} = (1 + \tau^C) p^h$ and $p^{C,m} = (1 + \tau^C) p^m$. Note that, because of the linear homogeneity of the aggregator, the consumption price index p^C can be rewritten by pulling out the taxation factor $(1 + \tau^C)$.

$$p^C = (1 + \tau^C) \tilde{p}^C, \quad \tilde{p}^C = \left[\xi^C \left(\tilde{p}^{C,m} \right)^{1-\lambda^C} + (1 - \xi^C) \left(\tilde{p}^{C,h} \right)^{1-\lambda^C} \right]^{\frac{1}{1-\lambda^C}}. \tag{97}$$

2.10.2 Other demands

We assume the exact same demand split procedure for investment and government consumption by using the same functional forms for investment technology and quasi-preferences for public consumption, but allowing for different import share and elasticity of substitution parameters. In contrast to investment demand, we assume that government consumption is taxed, similarly to private consumption, at the rate τ^{C^G}. The notation of prices works equivalently as for consumption, i.e., p^I is the price of the composite private investment good, p^{C^G} is the price for composite government consumption, and p^{I^G} is the price for the public investment good.

2.10.3 Export demand

Export demand is given by a reduced-form, partial equilibrium, downward-sloping export demand function for the domestic final good composite. This captures the idea that domestic and foreign goods are also perceived as imperfect substitutes by the foreign agents (which we did not explicitly model). If p^h rises above p^m, then demand for domestic goods from abroad shrinks but does not collapse to zero.[28] We assume

$$E^h = E\left(p^h\right), \tag{98}$$

which is a decreasing function. In the limiting case of infinitely elastic foreign demand, p^h stays constant, and any excess quantity of exports is absorbed abroad.

2.11 Production

Domestic production is carried out by atomistic firms, each producing a differentiated variety and exploiting monopoly power. They produce outputs Y_i^h using labor and capital, which they rent on competitive factor markets. The mass of the variety producers is normalized to 1 and fixed over time. This implies that those firms can earn economic rents even in the long run due to persistent market imperfections. Capital is provided by a competitive representative capital goods firm, which builds up the economy-wide capital stock by investing retained earnings. A competitive final goods assembly firm combines all differentiated varieties into a composite domestic final good, Y^h.

2.11.1 Final Good Assembly Firm[29]

A competitive final goods assembling firm combines individual varieties subject to the following CES production function:

$$Y^h = \left[\int_0^1 (Y_i^h)^{\frac{\vartheta-1}{\vartheta}} di\right]^{\frac{\vartheta}{\vartheta-1}} \tag{99}$$

with $\vartheta > 1$. Cost minimization gives the demand for variety i as

$$Y_i^h = \left(\frac{p^h}{p_i^h}\right)^{\vartheta} Y^h, \quad \text{with } p^h = \left[\int_0^1 (p_i^h)^{1-\vartheta} di\right]^{1/(1-\vartheta)}, \tag{100}$$

as the price index, which is taken as given by the individual variety producers. Note that in the case of symmetry, i.e., all variety producers charge the same price $p_i^h = p_{j\neq i}^h$ (which will occur in equilibrium), we simply have $Y^h = Y_i^h$ and $p^h = p_i^h$.

[28] See Keuschnigg and Kohler (2002) for a rationalization of this form stemming from an optimizing foreign agent.

[29] We drop the time index throughout this and the next subsection for convenience because of the static nature of the final goods firms' problems.

2.11.2 Variety Producing Firms

A producer of variety i faces the following inverted demand function from (100):

$$p_i^h(Y_i^h) = p^h(Y^h/Y_i^h)^{1/\vartheta}, \tag{101}$$

where p^h and Y^h are taken as given because of the small size of a variety producer compared to the total market. A variety producer produces Y_i^h using a production function that is linearly homogeneous in private inputs.

$$Y_i^h = \Phi \cdot f\left(\left\{L_i^{D,s}\right\}_{s=1,\dots,S}, K_i\right) \tag{102}$$

where Φ_t is the economy-wide productivity level, which is determined by the public capital stock $\Phi_t = \Phi(K_t^G)$, and taken as given by individual variety producers. $L_i^{D,s}$ is the skill-specific composite labor demand of an individual variety producer, consisting of $L_i^{D,Y,s}$ and $L_i^{D,O,s}$ (see Section 2.11). Firms do not directly hire labor but have to post vacancies $V_i^{Y,s}$ and $V_i^{O,s}$ at a cost. Vacancies are conditioned on skill and labor type (Y vs. O), but search is undirected, like in the canonical Diamond-Mortensen-Pissarides models of search and matching, w.r.t. to age a. This means a firm is randomly matched with a worker of age a, with the probability proportional to the relative frequency of workers of age a in the pool of unemployed (of a certain skill and labor type). Recall that a worker's productivity and supplied hours differ by age. As the firms know the age composition of the unemployed labor force, they can compute how many vacancies to post to employ a desired volume of labor units in expectation. Due to the law of large numbers, this will also be the ex-post volume of employed labor units. We can therefore write $L_t^{k,s} = E_t^{k,s} \cdot \bar{\theta}^s \bar{\ell}^s$, where $\bar{\theta}$ and $\bar{\ell}$ are productivity and hours averaged over age. Now, combine this with the law of motion for the number of employed persons (9), rescaled to an individual firm, to derive the law of motion for labor units.

$$L_{t,i}^{k,s} = V_{t,i}^{k,s} q_t^{f,k,s} \bar{\theta}_t^s \bar{\ell}_t^s + L_{i,t-1}^{k,s} \cdot \iota, \tag{103}$$

with the rebasing factor for changes in labor force and average productivity and hour supply $\iota \equiv \frac{\bar{\theta}_t^s \bar{\ell}_t^s}{\bar{\theta}_{t-1}^s \bar{\ell}_{t-1}^s} \frac{LF_t^{s,k}}{LF_{t-1}^{s,k}}$, importantly, observe that $\partial L_i^{k,s} / \partial V_i^{k,s} = q^{f,k,s} \bar{\theta}^s \bar{\ell}^s$. A variety producer then faces the following optimization problem:

$$\begin{aligned}\Pi_i^h = &\max_{K_i,V_i^{k,s}} \ (1-\tau^{prof}) \Big[(1-\tau^Y)p_i^h(Y_i^h)Y_i^h - (1+\tau^K)p^K K_i - \\ &(1+\tau^F)\textstyle\sum_{k,s} w^{k,s}L_i^{k,s} - (1-sub^f)p^h \bar{f}_i - p^h \sum_{k,s} c^{k,s}V_i^{k,s}\Big], \end{aligned} \tag{104}$$

where c are vacancy posting costs, and $\bar{f}_i$ are fixed costs in terms of the composite final good, the latter subject to a proportional fixed cost subsidy sub^f. τ^F are labor taxes paid by the employer, τ^K represents taxes on capital inputs, and τ^Y is an output tax. Variety producers earn economic rents that are taxed at rate τ^{prof}, which, in contrast to capital accumulation, does not distort behavior. Let us denote revenue as $\Upsilon = p_i^h(Y_i^h)Y_i^h$. The first-order conditions imply the equalization of input unit costs and marginal revenue, i.e., $(1-\tau^Y)\Upsilon_{K_i} = (1+\tau^K)p^K$ and $(1-\tau^Y)\Upsilon_{L_i^{k,s}} = (1+\tau^F)w^{k,s} + c^{k,s}/(q^{f,k,s}\bar{\theta}^s\bar{\ell}^s)$. Marginal revenue w.r.t. capital is

$$\Upsilon_K = \frac{\partial Y_i^h}{\partial K_i}\left[p_i^h(Y_i^h) + Y_i^h \cdot \frac{\partial p_i^h}{\partial Y_i^h}\right] = \frac{\partial Y_i^h}{\partial K_i} \cdot p_i^h(Y_i^h) \cdot \left[1+\epsilon_y^p\right], \tag{105}$$

where the elasticity of the output price with respect to output is simply $\epsilon_y^p = -1/\vartheta$ from (101). Hence, $\mu = 1/(1+\epsilon_y^p) \geq 1$ is the markup over marginal costs. For the case $\vartheta \to \infty$, the model also nests perfect competition as a limiting case. The optimality conditions for capital, and analogously for labor, are therefore

$$\mu \cdot (1+\tau^K)p^K = (1-\tau^Y)Y_{K_i}^h \cdot p_i^h, \tag{106}$$

$$\mu \cdot \left[(1+\tau^F)w^{k,s} + \frac{p^h c^{k,s}}{q^{f,k,s}\bar{\theta}^s\bar{\ell}^s}\right] = (1-\tau^Y)Y_{L_i^{k,s}}^h \cdot p_i^h. \tag{107}$$

Insert the price of capital in steady state (126) into (106) to derive the user cost of capital.

$$Y_{K_i}^h/\mu = \frac{p^I}{p_i^h}\frac{(1+\tau^K)\left[(r^{V,h}+\delta^K)(1-sub^I) - \phi_0^\tau \tau^{prof}\delta^K\right]}{(1-\tau^Y)(1-\tau^{prof})}. \tag{108}$$

Wage determination

The model provides two options. First, as a limiting case $c \to 0$, there can be instantaneous matching, i.e., $q(\theta) \to 1$. In this case (107) is reduced to $\mu \cdot (1+\tau^F)w^{k,s} = (1-\tau^Y)Y_{L_i^{k,s}}^h \cdot p_i^h$, which, after aggregation, gives the typical labor demand schedule with the negative relationship of $L_t^{D,k,s}$ and $w_t^{k,s}$, which in equilibrium intersects the labor supply curve and will pin down the wage rate. In the second option, search frictions prevail, and (107) resembles the job creation curve of the canonical search and matching model, which pins down labor market tightness $\theta^{k,s}$, while the wage rate is the result of a bargaining game. We assume the following ad-hoc wage rate schedule as the outcome of this bargaining game with workers' bargaining weight $\omega \in (0,1)$[30].

$$w^{k,s} = (1-\omega)\left[\phi^u \bar{w}^{k,s}\right] + \omega\left[\frac{(1-\tau^Y)}{(1+\tau^F)} p^h Y^h_{L^{k,s}}/\mu\right], \tag{109}$$

where ϕ^u is the net replacement rate. Ex-post, this is

$$w^{k,s} = \frac{\omega}{1-(1-\omega)\phi^u}\left[\frac{(1-\tau^Y)}{(1+\tau^F)} p^h Y^h_{L^{k,s}}/\mu\right]. \tag{110}$$

Let $\hat{\omega} = \frac{\omega}{1-(1-\omega)\phi^u}$. After inserting (110) into (107), we get the job creation condition that pins down labor market tightness (with q^f as a function of it):

$$(1-\hat{\omega})\left[(1-\tau^Y)Y^h_{L^{k,s}}/\mu\right]\bar{\theta}^s\bar{\ell}^s = \frac{c^{k,s}}{q^{f,k,s}}, \tag{111}$$

where we already used the symmetry results $p^h_i = p^h$. Note that we can rearrange even more. Insert (110) again in (111) to get an equilibrium relationship of wage rate and labor market tightness[31].

$$(1+\tau^F)w^{k,s}\frac{1-\hat{\omega}}{\hat{\omega}}\bar{\theta}^s\bar{\ell}^s = \frac{p^h c^{k,s}}{q^{f,k,s}}, \tag{112}$$

which, after inserting in (107) to eliminate the vacancy costs, gives

$$\frac{\mu}{\hat{\omega}}(1+\tau^F)w^{k,s} = (1-\tau^Y)Y^h_{L_i^{k,s}} \cdot p^h_i, \tag{113}$$

which are handy expressions for the implementation.

[30] Compare this to the wage schedule derived from Nash bargaining in the canonical search and matching model with linear utility, a bargaining weight of ω, given hours and productivity such that $y^{lab} = w\bar{\ell}\bar{\theta}$, and enriched by taxes: $y^{lab} = (1-\omega)\left[\frac{b^u}{(1-\tau^W)}\right] + \omega\left[\frac{(1-\tau^Y)}{(1+\tau^F)}p^h Y^h_L \bar{\ell}\bar{\theta}/\mu + \frac{p^h c\theta}{(1+\tau^F)}\right]$. Inserting $b^u = \phi^u(1-\tau^W)w\bar{\ell}\bar{\theta}$, ignoring the continuation part $c\theta$, and dividing by productivity and hours would give the same formula for the wage rate as our ad-hoc schedule (110).

[31] Equation (112) can also be rearranged in terms of total vacancy posting costs: $(1+\tau^F)w^{k,s}\frac{1-\hat{\omega}}{\hat{\omega}}\bar{\theta}^s\bar{\ell}^s\bar{q}^{w,k,s}u^{k,s}_{t-1}LF^{k,s} = p^h c^{k,s}V^{k,s}$.

Total input demands

Total input demands are trivially equal to the individual input demands times the number of firms, which is equal to one.

$$K^D = K_i, \tag{114}$$

$$L^{D,j} = L_i^j, \quad \forall j \in \mathcal{S} \times \{Y, O\}. \tag{115}$$

Because of symmetry and the fixed unit mass of firms, the total value of output can be written as $p_i^h Y_i = p^h Y$. Using weighted averages, the value of total labor demand (wage sum) is $L_t^D w_t = \sum_j L_t^{D,j} w_t^j$. The total number of posted vacancies is $V_t = \sum_j V_t^j$. The aggregate firm value of all intermediate producers is positive if the economic rents exceed the fixed costs.

$$V_t^F = \Pi_t^h + \hat{\mathcal{G}}_t V_{t+1}^F / R_t^{V,h}. \tag{116}$$

Shares of final goods producers and the capital good firm are perfect substitutes, $V_t^h = V_t^C + V_t^F$.

2.11.3 Capital Goods Firm

Private capital is owned by the capital goods firm, which is owned by the households, and is assumed to be mobile within the economy. Final goods producers pay a competitive rental rate p^K. The capital goods firm uses final goods to produce capital, which generates the demand for private investment, and K_t is the private capital stock. The capital firms' per-period dividend is

$$\chi_t = p_t^K K_t - (1 - sub_t^I) p_t^I I_t - p_t^I J_t + sub_t^l - T_t^F, \tag{117}$$

where I_t is private investment, $J_t(I_t, K_t)$ are capital adjustment costs[32], which are normalized to zero in steady state.

$$J(I_t, K_t) = \frac{1}{2} \psi K_t \left(\frac{I_t}{K_t} - (\delta^K + \hat{g}) \right)^2. \tag{118}$$

[32] Note that gross investment is $I_t + J_t$, which represents total private investment demand for final goods. We combine both in our notation when looking at sublevel demands, e.g., $I_t^h = i_t^h \cdot (I_t + J_t)$, etc.

sub^I_t is a subsidy to investment while sub^l_t is a cash-flow subsidy which does not distort the investment decision. T^F are profit taxes paid by the capital goods firm which we assume to take the following form

$$T^F_t = \tau^{prof}_t \left[p^K_t K_t - p^I_t \phi^\tau_0 \left(J_t + \delta^K K_t \right) \right]. \tag{119}$$

The capital goods firm can deduct wage costs from the tax base. Further, the tax system parameter $\phi^\tau_0 \in \{0, 1\}$ controls whether a deduction of capital replacement investment from the tax base is allowed or not. Note that because investments are made from retained earnings, the "true" capital costs will always exceed the deductible part, as the opportunity costs of internal financing are assumed to be non-deductible.[33] Inserting for T^F gives a single expression for dividends χ.

$$\begin{aligned} \chi_t = {} & (1 - \tau^{prof}_t) p^K_t K_t - (1 - sub^I_t) p^I_t I_t - (1 - \phi^\tau_0 \tau^{prof}_t) p^I_t J_t \\ & + \phi^\tau_0 \tau^{prof} \delta^K K_t + sub^l_t. \end{aligned} \tag{120}$$

The value of the firm is the discounted stream of per-period profits, i.e.,

$$V^C_t = \chi_t + \frac{\hat{\mathcal{G}}_t V^C_{t+1}}{R^{V,h}_t}, \tag{121}$$

and the law of motion for private capital is

$$\hat{\mathcal{G}}_t K_{t+1} = (1 - \delta^K) K_t + I_t. \tag{122}$$

The firm solves the following problem:

$$V^C(K_t) = \max_{I_t} \ \chi_t + \frac{\hat{\mathcal{G}}_t V^C(K_{t+1})}{R^{V,h}_t}. \tag{123}$$

[33] As shown later, this implies that the tax distortion cannot be undone using the assumed method of deductibility.

Define the marginal benefit of an increase in capital as $q_t \equiv V'(K_t)$. The optimality and envelope conditions are

$$\begin{aligned} I_t: \quad q_{t+1} &= R_t^{V,h} p_t^I (1 - sub_t^I + (1 - \phi_0^\tau \tau_t^{prof}) J_{I_t}) && (124) \\ K_t: \quad q_t &= (1 - \tau_t^{prof}) p_t^K - (1 - \phi_0^\tau \tau_t^{prof}) p_t^I J_{K_t} \\ &+ \phi_0^\tau \tau_t^{prof} p_t^I \delta^K + \frac{q_{t+1}}{R_t^{V,h}} (1 - \delta^K). && (125) \end{aligned}$$

To see how the tax instruments work, let us for the moment focus on the steady state where $J = J_I = J_K = 0$. Combine (124) with (125) and rearrange to get

$$p^K = p^I \frac{(r^{V,h} + \delta^K)(1 - sub^I) - \phi_0^\tau \tau^{prof} \delta^K}{(1 - \tau^{prof})}, \quad (126)$$

where the right-hand side reflects the price of capital. Observe how, in the absence of any tax instrument, this would simply be $p^I(r^{V,h} + \delta^K)$. The profit tax rate clearly increases the price of capital and therefore reduces capital usage, while the opposite is true for an investment subsidy. In the case that $\phi_0^\tau = 1$ (and ignoring the investment subsidy for the moment), the tax system is not completely neutral, as the price of capital is reduced to $p^I \left[r^{V,h}/(1 - \tau^{prof}) + \delta^K \right]$, which is, however, lower than $p^I \left[(r^{V,h} + \delta^K)/(1 - \tau^{prof}) \right]$ in the case of $\phi_0^\tau = 0$.

We briefly establish Hayashi (1982)'s theorem, which connects the capital goods firm value V^C and the private capital stock K.

Theorem 2.1. *Hayashi's theorem. Firm value and capital stock fulfill the following relationship:*

$$q_t K_t = V_t^C - V_t^R, \quad \forall t \quad (127)$$

where the V_t^R is the discounted firm rent, i.e.

$$V_t^R = firmrent_t + \frac{\hat{\mathcal{G}}_t V_{t+1}^R}{R_t^{V,h}}, \quad (128)$$

Proof is provided in the appendix. The per-period firm rent in this setting is just the cash flow subsidy, i.e., $firmrent_t = sub_t^l$. Using the law of motion for capital (122) and Hayashi's theorem, evaluated at the optimality condition for investment, i.e.,

$q_{t+1} = R_t^{V,h} p_t^I [1 - sub_t^I + (1 - \phi_0^\tau \tau_t^{prof}) J_{I_t}]$, gives an implicit relation for investment:

$$I_t = \frac{\hat{\mathcal{G}}_t \left[V_{t+1}^C - V_{t+1}^R\right]}{R_t^{V,h} p_t^I (1 - sub_t^I + (1 - \phi_0^\tau \tau_t^{prof}) J_{I_t})} - (1 - \delta^K) K_t. \tag{129}$$

2.12 Government

The government sector is characterized by the following: First, it raises revenue according to the tax bases described above. Second, the government consumes domestic and foreign varieties. In contrast to households and firms, the level of composite public consumption and investment is not micro-founded but exogenously given by $C_t^G \geq 0$ and $I_t^G \geq 0$. Government consumption (already by its name) is assumed to be purely consumptive, in contrast to public investment, which builds up the public capital stock, e.g., infrastructure, which influences the economy's productivity. Third, the government provides social security by granting unemployment and welfare benefits to unemployed and non-participants. Fourth, the government can subsidize firms in different ways and can give lump-sum transfers to households. Fifth, the government issues debt in the form of government bonds. Per period, government expenditure is given by

$$Exp_t = p_t^{C^G} \cdot C_t^G + p_t^{I^G} \cdot I_t^G + B_t + P_t + Sub_t^I + Sub_t^l + Trans_t + Z_t^{gov}, \quad \text{where} \tag{130}$$

$$
\begin{aligned}
C_t^G &= c_t^G N_t + C_t^{G,A} + C_t^{G,exo}, \quad \text{(government consumption)} \\
C_t^{G,A} &= \sum_{i \in \mathcal{A} \times \mathcal{S}} c_t^{G,A,i} N_t^i, \quad \text{(age-skill-specific part in } C^G\text{)} \\
I_t^G &= \text{ public investment} \\
B_t^u &= \sum_{i \in \mathcal{A} \times \mathcal{S}} \delta_t^i u_t^i b_t^{u,i} N_t^i, \quad \text{(unemployment benefits)} \\
B_t^n &= \sum_{i \in \mathcal{A} \times \mathcal{S}} (1 - \delta_t^i) \phi_t^i b_t^{n,i} N_t^i, \quad \text{(non-participation benefits)} \\
B_t^k &= \sum_{i \in \mathcal{A} \times \mathcal{S}} \delta_t^i b_t^{k,i} N_t^i, \quad \text{(participation benefits)} \\
B_t &= B_t^n + B_t^u + B_t^k, \quad \text{(total non-employment benefits)} \\
P_t &= \sum_{i \in \mathcal{A} \times \mathcal{S}} (1 - \delta_t^i)(1 - \phi_t^i) y_{pens,t}^i N_t^i, \quad \text{(pensions)} \\
Sub_t^I &= sub_t^I \cdot p_t^I I_t, \quad \text{(investment subsidy)} \\
Sub_t^l &= sub_t^l, \quad \text{(cash flow subsidy)} \\
Sub_t^f &= sub_t^f \cdot p_t^h \bar{f}_t, \quad \text{(fixed costs subsidy)} \\
Trans_t &= p_t^h \sum_i \tau^{-,i}, \quad \text{(lump-sum transfers)} \\
Z_t^{gov} &= \text{ public capital transfer to abroad.}
\end{aligned}
$$

Revenues are

$$
Rev_t = T_t^L + T_t^T + T_t^R + T_t^C + T_t^Y + T_t^K + T_t^{prof}, \quad \text{where} \tag{131}
$$

$$
\begin{aligned}
T_t^L &= \left(\tau_t^F L_t^D + \tau_t^W L_t^S\right) w_t, \quad \text{(labor taxation)} \\
T_t^P &= \sum_{i \in \mathcal{A} \times \mathcal{S}} (1-\delta_t^i)(1-\phi_t^i)\tau_t^{W,i} y_{pens,t}^i N_t^i, \quad \text{(pension taxation)} \\
T_t^T &= \tilde{p}_t^C \sum_{i \in \mathcal{A} \times \mathcal{S}} \tau_t^{+,i} N_t^i, \quad \text{(lump-sum taxation)} \\
T_t^R &= \tau_t^R \cdot i_t^W \frac{S_t}{1+i_t^W}, \quad \text{(interest taxation)} \\
T_t^C &= \tau_t^C \cdot \tilde{p}_t^C C_t + \tau_t^{C^G} \cdot \tilde{p}_t^{C^G} C_t^G, \quad \text{(consumption taxation)} \\
T_t^Y &= \tau_t^Y \cdot p_t^h Y_t^h, \quad \text{(output taxation)} \\
T_t^K &= \tau_t^K \cdot p_t^K K_t, \quad \text{(capital taxation)} \\
T_t^{prof} &= \tau_t^{prof} \Bigg[p_t^h \left((1-\tau_t^Y) Y_t^h - (1-sub_t^f)\bar{f}_t - cV_t \right) - \tau_t^K p_t^K K_t - \\
&\qquad (1+\tau_t^F) w_t L_t^D - \phi_0^\tau p_t^I (J_t + \delta^K K_t) \Bigg]. \quad \text{(profit taxation)}
\end{aligned}
$$

$\tau_t^{C^G}$ is the effective tax rate on government consumption, which is typically smaller than for private consumption as parts of public consumption are tax exempted. Economically, taxing government consumption has no effect (like putting money from one into the other pocket) as it cancels in the primary balance, but modeling it this way helps to match empirical revenue/expenditure data. Government consumption can consist of different types, i.e., a part that is proportional to total population, a part that depends on age and skill-specific profiles (e.g., health care, long-term care (LTC) or education) and a part that is independent of population size and composition. Similarly, lump-sum net taxes (denoted τ^l in the household problem) can be split by type, first into lump-sum gross taxes τ^+ and lump-sum transfers τ^- (i.e., $\tau^l = \tau^+ - \tau^-$), which themselves can consist of different parts (family benefits, LTC cash benefits, etc.) with different age-skill profiles. Timing is crucial for interest taxation as interest is earned at the end of period, while all other flows including taxation are paid at the beginning of period. Therefore the interest tax payment has to be discounted with $1/R_t^W$ in order to be consistent. This can be derived by rearranging the aggregate asset equation (71)

$$\begin{aligned}\hat{\mathcal{G}}_t A_{t+1} = \bar{R}_t^W S_t \;\Leftrightarrow\; \hat{\mathcal{G}}_t A_{t+1} &= \frac{1 + i_t^W(1-\tau_t^R)}{\mathcal{G}_t^{€}} S_t \;\Leftrightarrow \\ \hat{\mathcal{G}}_t A_{t+1} &= R_t^W S_t - \tau_t^R i_t^W / \mathcal{G}_t^{€} S_t \;\Leftrightarrow \\ \hat{\mathcal{G}}_t A_{t+1}/R_t^W &= S_t - \tau_t^R i_t/(1+i_t^W) S_t = S_t - T_t^R = \hat{A}_t,\end{aligned}$$

where the interest taxation term is measured at the same point in time as the other flows in S_t. The overall budget can be divided into the pension system and the general budget. Payments into the pension system are the sum of all contributions, i.e., $\left(\tau_t^{F,p} L_t^D + \tau_t^{F,p} L_t^S\right) w_t$, while the expenditure is given by $\sum_{i\in\mathcal{A}\times\mathcal{S}}(1-\delta_t^i)(1-\phi_t^i) y_{pens,t}^i N_t^i$. If contributions fall short of the pension payouts, the system is implicitly cross-subsidized by the general budget. The primary balance is defined as

$$PB_t = Rev_t - Exp_t, \tag{132}$$

which changes the stock of public debt[34] according to

$$\hat{\mathcal{G}}_t D_{t+1}^G = R_t^{G,h}\left[D_t^G - PB_t\right]. \tag{133}$$

Public expenditure on interest is $Exp_t^R = i_t^{G,h}\left[D_t^G - PB_t\right]$. One has to set at least one budget rule, namely, one that targets long-run debt at a level[35] that is bounded from above as well as below.

$$\exists B : \lim_{t\to\infty} D_t^G = B \text{ and } |B| < \infty, \tag{134}$$

otherwise, no final steady state would exist. The budget rule has to involve at least one government instrument, as well as a feasible path for the evolution of D^G. Note that, for example, big immediate downward jumps, e.g., $D_t^G - \hat{\mathcal{G}}_t D_{t+1}^G \gg 0$, might prove to be infeasible if they imply such a severe tax increase that it is simply not achievable (breakdown of equilibrium). An example of a budget rule starting from D_0^G would be to adjust employees' wage taxes τ_t^W every period in order to have $D_t^G = D_0^G$, $\forall t > 0$. Another would be to fix all instruments for 20 years and let D^G

[34]Note that in the data, public debt is typically recorded at the end of the period. So, in retrended terms, we have $\tilde{D}_t^{G,end} = \tilde{D}_{t+1}^G$ and usually report the end-of-period value in the results. In detrended terms, this translates to $D_t^{G,end} = \hat{\mathcal{G}}_t \mathcal{G}_t^{€} D_{t+1}^G$.

[35]Note that in a model with exogenous growth, one has to target a bounded level of government debt in detrended terms.

change freely, and then use changes in government consumption to fix government debt at its current level, i.e., $D^G_t = D^G_{t=20}$, $\forall t > 20$. The budget rule pins down PB_t (at least at some point). For interpretation purposes, it can often be convenient to split the primary balance into a no-policy-change ("npc") and an adjustment part, $PB_t = PB^{npc}_t + PB^{adjust}_t$. The no-policy-change part is then computed by freezing the budget instrument corresponding with the chosen budget rule after some t. Then PB^{adjust} measures the required adjustment every year starting in t to comply with the budget rule.[36]

In case of infinitely elastic foreign demand for government bonds, one can treat $i^{G,h}_t$ as exogenous. When $i^{G,h}_t$ is exogenous, we can interpret it as the average rate that stems from a simple debt roll-over mechanic without explicitly modeling bonds of varying maturity as separate asset types. Let $\tilde{i}^{G,h}_t$ be the current market rate, and the yearly redemption employing a simple hazard rate $Red_t = o_t \left[DG_t - Exp^R_{t-1}/\hat{\mathcal{G}}_{t-1}\right]$. We can then write the evolution of the average nominal interest rate as

$$\begin{aligned} i^{G,h}_t = & \frac{(1-o_t)\left[D^G_t - Exp^R_{t-1}/\hat{\mathcal{G}}_{t-1}\right]}{D^G_t - PB_t} \cdot i^{G,h}_{t-1} \\ & + \frac{o_t DG_t - PB_t + (1-o_t)Exp^R_{t-1}/\hat{\mathcal{G}}_{t-1}}{D^G_t - PB_t} \cdot \tilde{i}^{G,h}_t. \end{aligned} \tag{135}$$

Note that in the case of $o_t = 1$, this collapses to $i^{G,h}_t = \tilde{i}^{G,h}_t$. With $o_t < 1$, changes in the current rate feed into changes in the average rate only sluggishly.

The law of motion for the public capital stock is given as

$$K^G_{t+1} = (1-\delta^G)K^G_t + (I^G_t/VA_t). \tag{136}$$

The current public capital stock maps into a productivity index Φ_t. To ensure balanced growth, we assume the productivity of the public capital stock is independent of labor-augmenting and population growth. The efficiency of the public capital stock is therefore increased only if a higher share of public investment, in terms of total value added, is spent, where value added is $VA_t = p^h_t(Y^h_t - \bar{f}_t - cV_t)$. This setting implies that a permanent increase in the public investment to value-added share will increase the output growth rate only in transition before output in the

[36]See section 5.6 on indicators that are based on PB^{adjust}_t.

long run again grows at rate g, but from a higher level. Supporting recent empirical evidence is, for example, provided by Gemmell et al. (2016) (see Section 3.6.3).

2.13 Market Clearing

In equilibrium, the following excess demands[37] have to be zero, and further conditions must hold.

$$
\begin{aligned}
L^s:\ \zeta_t^{L,j} &= L_t^{D,j} - L_t^{S,j} = 0,\ \ \forall j \in \mathcal{S} \times \{Y, O\} \ \Rightarrow\ w_t^j \text{ or } \theta_t^j && (137)\\
K:\ \zeta_t^{K} &= K_t^{D} - K_t = 0,\ \ \Rightarrow\ p_t^K, && (138)\\
Y:\ \zeta_t^{Y} &= C_t^h + I_t^h + C_t^{G,h} + I_t^{G,h} + E_t^h - \hat{Y}_t^h = 0,\ \Rightarrow\ p_t^h, && (139)\\
V^h:\ \zeta_t^{V} &= \hat{V}_t^h - \hat{A}_t^{V,h} - \hat{A}_t^{*V,h} = 0,\ \ \Rightarrow\ r_t^{V,h} && (140)\\
D^G:\ \zeta_t^{DG} &= \hat{D}_t^G - \hat{A}_t^{G,h} - \hat{A}_t^{*G,h} = 0,\ \ \Rightarrow\ r_t^{G,h} && (141)\\
PB:\ PB_t &= \text{target according to the chosen budget rule.} && (142)
\end{aligned}
$$

We defined net output as $\hat{Y}_t^h = Y_t^h - \bar{f}_t - cV_t$. Note that while (137) indicates labor market clearing, this is actually not the case with search frictions, as there is involuntary unemployment. $\hat{A}^{*V,h}$ and $\hat{A}^{*G,h}$ denote foreign demand for domestic assets. We assume reduced-form demand functions that non-negatively depend on the corresponding return rates, i.e., $r^{V,h}$ and $r^{G,h}$, respectively. The model nests the specification of infinitely elastic demand functions such that either one or both, $r^{V,h}$ and $r^{G,h}$, are exogenously fixed, and foreign demand absorbs all excess supply of the two domestic assets. We can collect and write all dynamic (detrended) asset equations as

$$
\begin{aligned}
\hat{\mathcal{G}}_t D_{t+1}^G &= R_t^{G,h}\left[D_t^G - PB_t\right] = R_t^{G,h} \cdot \hat{D}_t^G, && (143)\\
\hat{\mathcal{G}}_t D_{t+1}^F &= R_t^{D}\left[D_t^F + TB_t + A_t^{Mig} - Z_t - Z_t^{gov}\right] = R_t^D \cdot \hat{D}_t^F,\ \text{ with} && (144)\\
& \qquad TB_t = p_t^{E,h} E_t^h - \sum_z \tilde{p}_t^{z,m} z_t^m,\ z \in \left\{C, I, C^G, I^G\right\} && (145)\\
\hat{\mathcal{G}}_t A_{t+1} &= \bar{R}_t^W\left[A_t + A_t^{Mig} - Z_t + y_t - p_t^C C_t\right] = R^W \cdot \hat{A}_t, && (146)\\
\hat{\mathcal{G}}_t V_{t+1} &= R_t^{V,h}\left[V_t - \chi_t\right] = R_t^{V,h} \cdot \hat{V}_t, && (147)\\
\hat{\mathcal{G}}_t V_{t+1}^F &= R_t^{V,h}\left[V_t^F - n_t^h \Pi_{i,t}\right] = R_t^{V,h} \cdot \hat{V}_t^F. && (148)
\end{aligned}
$$

[37]Note that the individual variety markets hold by construction and are therefore skipped in the derivation of Walras' Law.

Note that consumption taxes paid on imported goods are not part of the trade balance. It is convenient to define $\hat{V}_t^h \equiv \hat{V}_t^C + \hat{V}_t^F$ as the sum of all end-of-period domestic firm assets, and analogously $V_t^h \equiv V_t^C + V_t^F$ for beginning-of-period domestic firm assets. The average return rate r^D of foreign assets is derived as follows. The right-hand side of (144) can be rewritten as

$$R_t^D \cdot \hat{D}_t^F = \hat{D}_t^F + (r_t^{V,m}\hat{A}_t^{V,m} - r_t^{V,h}\hat{A}_t^{*V,h}) + (r_t^{G,m}\hat{A}_t^{G,m} - r_t^{G,h}\hat{A}_t^{*G,h}), \tag{149}$$

$$\text{with } \hat{D}_t^F = (\hat{A}_t^{V,m} - \hat{A}_t^{*V,h}) + (\hat{A}_t^{G,m} - \hat{A}_t^{*G,h}). \tag{150}$$

The average relevant rate of return of foreign assets r^D is, therefore,

$$r_t^D = \frac{(r_t^{V,m}\hat{A}_t^{V,m} - r_t^{V,h}\hat{A}_t^{*V,h}) + (r_t^{G,m}\hat{A}_t^{G,m} - r_t^{G,h}\hat{A}_t^{*G,h})}{(\hat{A}_t^{V,m} - \hat{A}_t^{*V,h}) + (\hat{A}_t^{G,m} - \hat{A}_t^{*G,h})}. \tag{151}$$

2.13.1 Limiting cases: Closed economy and semi-open economy

The model is mainly designed to capture the characteristics of a small, open economy, but also nests a closed economy version as a limiting case.

Definition 2.1. ***The closed economy*** *is nested if the following conditions are met:*

a) all import shares are zero: $\xi^i = 0,\ \forall i \in \{C, C^G, I, I^G\}$,

b) export demand is zero: $E^h = 0$,

c) transfers abroad and assets brought by net migrants are zero: $Z = A^{Mig} = 0$,

d) foreign demand for domestic assets is zero: $\hat{A}^{*V,h} = \hat{A}^{*G,h} = 0$, *and*

e) $r^{V,h}$ *and* $r^{G,h}$ *are the varying prices that clear asset markets while* p^h *is constant (numéraire).*

A consequence of these conditions is that $TB = DF = 0$. Note that in a closed economy, capital adjustment costs are no longer required (one can set $\psi = 0$). We also allow for the special case of a semi-open economy.

Definition 2.2. ***The semi-open economy*** *is a special case of the open economy setting where instead of fixing interest rates, we fix a path of foreign asset positions* DF *(and therefore the current account* TB*) and set* $r^{V,h}$ *in order to replicate this* DF*-path in equilibrium.*

2.14 Steady State

Most of the steady-state equations are trivial to derive: we simply have to drop the time index of the static relationships from Section (2.13). We therefore just report those relationships that involve some algebraic steps. Importantly, note that in steady state, the following is in general **not true**: $X_t^a = X_{t+1}^{a+1}$ for some variable X.

The steady-state values of the asset components are

$$V^C = \chi \cdot \frac{R^{V,h}}{r^{V,h} - \hat{g}}, \quad D^F = -\left[TB + A^{Mig} - Z - Z^{gov}\right] \cdot \frac{R^D}{r^D - \hat{g}}, \quad D^G = PB \cdot \frac{R^{G,h}}{r^{G,h} - \hat{g}}.$$

2.15 Walras' Law

Next to (137) to (152), define the following additional excess demands:

$$\begin{aligned}
\text{aggregate assets}: &\quad \zeta_t^{A} = V_t^h + D_t^F + D_t^G - A_t, && (152)\\
\text{foreign assets}: &\quad \zeta_t^{DF} = \hat{D}_t^F - \hat{A}_t^{V,m} + \hat{A}_t^{*V,h} - \hat{A}_t^{G,m} + \hat{A}_t^{*G,h}, && (153)\\
\text{government}: &\quad \zeta_t^{G} = Rev_t - Exp_t - PB_t && (154)\\
\text{iter-vivo transfers}: &\quad \zeta_t^{IV} = -iv_t && (155)\\
\text{accidental bequest}: &\quad \zeta_t^{AB} = \sum_{s=1}^{S} \sum_{a=\underline{a}}^{\overline{a}} (1 - \gamma_t^{a,s}) S_t^{a,s} N_t^{U,a,s} - ab_t && (156)
\end{aligned}$$

Theorem 2.2. *Walras' Law. The following two conditions have to hold for all prices and all values of the chosen budget-closing instrument, i.e., even out of equilibrium:*

***Dynamic version of Walras' Law**:*

$$\begin{aligned}
&p_t^h \zeta_t^Y + p_t^K \zeta_t^K + \sum_j w_t^j \zeta_t^{L,j} + \zeta_t^G + \zeta_t^{IV} + \zeta_t^{AB} \\
&+ \left[R_{t-1}^D \zeta_{t-1}^{DF} + R_{t-1}^{G,h} \zeta_{t-1}^{DG} + R_{t-1}^{V,h} \zeta_{t-1}^{V}\right] / \hat{\mathcal{G}}_{t-1} - \left[\zeta_t^{DF} + \zeta_t^{DG} + \zeta_t^{V}\right] = 0.
\end{aligned} \quad (157)$$

***Steady state version of Walras' Law**:*

$$\begin{aligned}
&p^h \zeta^Y + p^K \zeta^K + \sum_j w^j \zeta^{L,j} + \zeta^G + \zeta^{IV} + \zeta^{AB} \\
&+ \frac{r^{V,h} - \hat{g}}{\hat{\mathcal{G}}} \zeta^V + \frac{r^{G,h} - \hat{g}}{\hat{\mathcal{G}}} \zeta^{DG} + \frac{r^D - \hat{g}}{\hat{\mathcal{G}}} \zeta^{DF} = 0.
\end{aligned} \quad (158)$$

Proof is provided in the appendix.

3 Functional Forms

3.1 Utility

We incorporate two types of utility functions: one with and one without income effects on labor supply. All utility specifications allow for balanced growth, though for the non-income-effects case, the disutility of labor must be assumed to grow with productivity. Let $i \in \mathcal{A} \times \mathcal{S} \times \mathcal{K}$ index a household of a specific age, skill, and savings type. The cohort index z is dropped for convenience. Household utility is scaled by a household size shifter[38] ω^i that is defined as

$$\omega^i = \left(1 + \omega_0^\sigma (n^i)^{1-\sigma+\sigma\omega_1}\right)^{1/\sigma}, \tag{159}$$

where n^i is the average number of dependent children ($a < \overline{a}$) in a household.[39]

With income effects

$$u(\tilde{C}^i, \Psi^i) = \omega^i \times \begin{cases} \frac{\left[\tilde{C}^i \cdot e^{-\Psi^i}\right]^{1-1/\sigma} - 1}{1-1/\sigma}, & \text{if } \sigma \neq 1. \\ \ln(\tilde{C}^i) - \Psi^i, & \text{if } \sigma = 1. \end{cases} \tag{160}$$

Marginal utilities are given by

$$u_{C^i} = \omega^i \cdot \left[\tilde{C}^i \cdot e^{-\Psi^i}\right]^{-1/\sigma} \cdot e^{-\Psi^i}, \tag{161}$$

$$u_{\delta^i} = \omega^i \cdot \left[\tilde{C}^i \cdot e^{-\Psi^i}\right]^{-1/\sigma} \cdot e^{-\Psi^i} \cdot \tilde{C}^i \cdot (-\Psi_{\delta^i}), \tag{162}$$

$$u_{s^i} = \omega^i \cdot \left[\tilde{C}^i \cdot e^{-\Psi^i}\right]^{-1/\sigma} \cdot e^{-\Psi^i} \cdot \tilde{C}^i \cdot (-\Psi_{s^i}), \tag{163}$$

$$u_{\ell^i} = \omega^i \cdot \left[\tilde{C}^i \cdot e^{-\Psi^i}\right]^{-1/\sigma} \cdot e^{-\Psi^i} \cdot \tilde{C}^i \cdot (-\Psi_{\ell^i}). \tag{164}$$

[38] In the implementation, the shifter will only differ by age. Note that $u(\tilde{C}^i, \Psi^i)$ is short for $u(\tilde{C}^i, \Psi^i; \omega^i)$, as ω^i is not a choice variable but implies that the functional form of $u(\cdot)$ is age-dependent.

[39] The weight can be microfounded from a Barro-Becker type problem of the form $\max_{C^P, C^C} u(C^P) + \omega_0 n^{\omega_1} u(C^C)$, s.t. $C^P + nC^C = C$, where C^P is the consumption of the parent, and C^C is the consumption of a dependent child. See, e.g., Auclert et al. (2021) for a proof that generalizes to the chosen functional forms in our model, if one assumes that children share the disutility from work of the parents.

Without income effects

$$u(\tilde{C}^i, \Psi^i) = \omega^i \times \begin{cases} \frac{\left[\tilde{C}^i - \Psi^i\right]^{1-1/\sigma} - 1}{1-1/\sigma}, & \text{if } \sigma \neq 1. \\ \ln(\tilde{C}^i - \Psi^i), & \text{if } \sigma = 1. \end{cases} \tag{165}$$

Marginal utilities are given by

$$u_{C^i} = \omega^i \cdot \left[\tilde{C}^i - \Psi^i\right]^{-1/\sigma}, \tag{166}$$

$$u_{\delta^i} = \omega^i \cdot \left[\tilde{C}^i - \Psi^i\right]^{-1/\sigma} \cdot (-\Psi_{\delta^i}), \tag{167}$$

$$u_{s^i} = \omega^i \cdot \left[\tilde{C}^i - \Psi^i\right]^{-1/\sigma} \cdot (-\Psi_{s^i}), \tag{168}$$

$$u_{\ell^i} = \omega^i \cdot \left[\tilde{C}^i - \Psi^i\right]^{-1/\sigma} \cdot (-\Psi_{\ell^i}). \tag{169}$$

The marginal disutilities of labor are given as[40]

$$\Psi_{\delta^i} = \varphi^{\delta\prime}(\delta^i) + u^{i-1}\varphi^s(s^i) + e^i\varphi^\ell(\ell^i) - (1-\phi^i)\varphi^{R\prime}(v^i) \tag{170}$$

$$\Psi_{s^i} = \delta^i u^{i-1}\varphi^{s\prime}(s^i) + \delta^i u^{i-1} q(\theta)\varphi^\ell(\ell), \tag{171}$$

$$\Psi_{\ell^i} = \delta^i e^i \varphi^{\ell\prime}(\ell^i). \tag{172}$$

The next section elaborates more on the functional form assumptions of the various $\varphi()$ functions.

3.2 Labor Supply

Intensive margin

First, we have to fix a functional form for the disutility function of hours, $\varphi^\ell(\cdot)$, and drop all skill, age, and cohort indices in the notation for convenience. We choose the following form:

$$\varphi^\ell(\ell) = \varphi_0^\ell \frac{\varepsilon_\ell}{1+\varepsilon_\ell} (\ell)^{\frac{1+\varepsilon_\ell}{\varepsilon_\ell}} - \varphi_1^\ell. \tag{173}$$

Computing the first derivative and inserting it into the first-order condition (46) reveals that

$$\ell = \left(\frac{(1-\hat{\tau}^\ell) w\theta / (\tilde{p}^C \Xi)}{\varphi_0^\ell}\right)^{\varepsilon_\ell}. \tag{174}$$

[40] Superscript $i-1$ is lazy notation and just refers to the last period; other than that, it refers to the same household characteristics, such as skill level, etc.

This implies that the Frisch elasticity of labor supply at the intensive margin with respect to the wage rate or the productivity parameter θ is

$$\frac{\partial \ln \ell}{\partial \ln w}_{|d\hat{\tau}^\ell/dw=0} = \frac{\partial \ln \ell}{\partial \ln \theta}_{|d\hat{\tau}^\ell/d\theta=0} = \varepsilon_\ell, \tag{175}$$

i.e., a 1% increase in the wage rate leads to an ε_ℓ% increase in the individual hours choice. ε_ℓ therefore represents the micro-elasticity of labor supply because it governs the individual decision. In contrast, if, e.g., θ is increased by 1% for all workers, the effect on average ℓ will not necessarily be equal to ε_ℓ. In this case, we are looking for the macro-elasticity of labor supply[41], which also includes general equilibrium effects through changes in the wage rate. This elasticity is usually inferred from shocking the model and computing the change in average ℓ. For models with simple production (e.g., like the non-nested CES production functions presented earlier), we can at least compute the long-run macro-elasticity, which in this case coincides with ε_ℓ. As a response to the productivity shock, wages drop only in the beginning but recover to their initial value once the capital stock grows to its new steady state level. However, the short-run macro-elasticity will be smaller. The parameter φ_0^ℓ is a scaling parameter, i.e., to target a value of ℓ in the calibration. φ_1^ℓ can be used to normalize the disutility costs $\varphi(\ell)$.

Search intensity margin

Disutility of search effort has the same functional form as hours, i.e.,

$$\varphi^s(s) = \varphi_0^s \frac{\varepsilon_s}{1+\varepsilon_s} (s)^{\frac{1+\varepsilon_s}{\varepsilon_s}} - \varphi_1^s. \tag{176}$$

Computing the first derivative and inserting it into the first-order condition (49) allows us to explicitly solve for s.

$$s = \left(q(\theta) \frac{(1-\hat{\tau}^s) w\ell\theta/(\tilde{p}^C \Xi) - \varphi^\ell(\ell)}{\varphi_0^s} \right)^{\varepsilon_s}. \tag{177}$$

[41] In addition to the distinction between micro and macro elasticities, one differentiates between Hicksian (constant utility), Marshallian (constant wealth), and Frisch (constant marginal utility of wealth) labor supply elasticities. In the non-income-effects specification, all three concepts coincide. See e.g., Chetty (2012) for a thorough discussion.

The Frisch elasticity of search effort with respect to the wage rate, or the productivity parameter, is

$$\frac{\partial \ln s}{\partial \ln w}_{|\varphi^{\ell}(\ell)=0,\ d\hat{\tau}^s/dw=0} = \frac{\partial \ln s}{\partial \ln \theta}_{|\varphi^{\ell}(\ell)=0,\ d\hat{\tau}^s/d\theta=0} = \varepsilon_s. \tag{178}$$

i.e., a 1% increase in the wage rate leads to an ε_s% increase in the individual search effort. φ_0^s and φ_1^s have the same function as their hours-supply counterparts.

Extensive margin and retirement

For the participation margin, it is convenient to assume the following functional form for disutility:

$$\varphi^{\delta}(\delta) = \varphi_0^{\delta}\varepsilon_{\delta} \cdot \exp\left(\frac{\delta}{\varepsilon_{\delta}}\right) - \varphi_1^{\delta} - \varphi_2^{\delta} \cdot \delta. \tag{179}$$

The advantage of this formulation is that δ can be calibrated to be 0. φ_0^{δ} is set to target the empirical participation rates, φ_1^{δ} can be chosen to normalize disutility (e.g., to 0) in calibration, and φ_2^{δ} is a shift parameter that helps for numerical stability. The functional form for the utility of retiring is

$$\varphi^R(\upsilon) = \upsilon \cdot \upsilon_0 LE^{\upsilon_1}, \quad \text{with } \upsilon_0 > 0, \tag{180}$$

which increases in the remaining life expectancy. Recall that the retirement share is $\upsilon = (1-\phi)(1-\delta)$; hence, we have $\partial\varphi^R(\upsilon)/\partial\delta = -(1-\phi)\varphi^{R\prime}(\upsilon)$, and the marginal utility is independent of δ. Using these functional forms in the first-order condition (53) gives the following solution for δ:

$$\delta = \varepsilon_{\delta} \cdot \ln\left(\frac{(1-\hat{\tau}^{\delta})ew\theta\ell/(\tilde{p}^C\Xi) - e\varphi^{\ell}(\ell) - u^{a-1}\varphi^s(s) - (1-\phi)\varphi^{R\prime}(\upsilon) + \varphi_2^{\delta}}{\varphi_0^{\delta}}\right). \tag{181}$$

This implies that the Frisch semi-elasticity of labor supply at the extensive margin with respect to the wage rate or the productivity parameter – ignoring $\varphi^{\ell}(\ell^a)$, $(1-\phi)\varphi^{R\prime}(\upsilon)$, the shift parameter φ_2^{δ}, and the fact that the participation tax rate varies with w or θ for the moment – would be simply given by the parameter ε^{δ}, i.e.,

$$\frac{\partial \delta}{\partial \ln w}_{|\bar{\varphi}=0,\ d\hat{\tau}^{\delta}/dw=0} = \frac{\partial \delta}{\partial \ln \theta}_{|\bar{\varphi}=0,\ d\hat{\tau}^{\delta}/d\theta=0} = \varepsilon_{\delta}, \tag{182}$$

with $\bar{\varphi} \equiv \varphi^{\ell}(\ell) + (1-\phi)\varphi^{R\prime}(\upsilon) - \varphi_2^{\delta}$. However, taking named factors into account implies that the actual semi-elasticity $\hat{\varepsilon}^{\delta}$ will generally differ from ε^{δ}, depending on age and skill.

Young labor share function

As a flexible specification for $\mu(\cdot)$, which is in line with the conditions set out in Section 2.6, we use a normalized incomplete beta function with support scaled to $[\underline{a}, \overline{a}]$ and evaluated at discrete $a \in \{\underline{a}, \ldots, \overline{a}\}$. More precisely, we allow that the share of young labor reaches 0 already at age $\tilde{a} \leq \overline{a}$, e.g., around the age of retirement, rather than the maximum attainable age $\overline{a}$. The specification reads

$$\mu(a) = \begin{cases} 1 - \frac{B(\hat{a};\beta_1,\beta_2)}{B(1;\beta_1,\beta_2)}, & \text{if } \underline{a} \leq a < \tilde{a}, \\ 0, & \text{if } \tilde{a} \leq a \leq \overline{a}, \end{cases} \tag{183}$$

where $\hat{a} = \frac{a-\underline{a}}{\tilde{a}-\underline{a}}$ and $B(x;\beta_1,\beta_2) = \int_0^x y^{\beta_1-1}(1-y)^{\beta_2-1}dy$. Shape parameters β_1 and β_2 govern the curvature of the weight transition in age. For example, $\beta_1 = \beta_2 = 1$ implies a linear increase in age.

3.3 Matching Function

We assume a labor market matching function of simple Cobb-Douglas form.

$$\mathcal{M}(\bar{s}_t u_{t-1} LF_t, V_t) = \mathcal{M}_0 \left(\bar{s}_t u_{t-1} LF_t\right)^{\eta} \left(V_t\right)^{1-\eta}, \tag{184}$$

with scaling factor $\mathcal{M}_0$ and elasticity η. Consequently, we have $q(\theta_t) = \mathcal{M}_0 \theta_t^{1-\eta}$ and $q^f(\theta_t) = \mathcal{M}_0 \theta_t^{\eta}$.

3.4 Tax Functions

In Section 2.6, we simply used $\tau_t^{W,a,s}$ as the average and $\tilde{\tau}_t^{W,a,s}$ as the marginal tax rate on labor income. They can be exogenously given, e.g., derived from age- and skill-specific tax payments according to microdata, or – by default – computed on the fly using the following tax functions that are independent of age and skill, which is why we drop the corresponding superscripts now (and the time index for convenience). Let $y^{lab} = \ell w \theta$ be gross labor income conditional on employment.

Then, net labor income is given by:

$$y^{lab} - T^p(y^{lab}) - T^c(y^{lab}) - T(y^{lab} - T^p(y^{lab}) - T^c_t(y^{lab})) \tag{185}$$

where T^p are pension contributions, T^c are other contributions (health care, unemployment insurance, etc.), and T is the paid income tax. For the first two tax functions, we assume proportional rates[42], i.e., $T^p(y^{lab}) = \tau^{W,p} \cdot y^{lab}$ and $T^c(y^{lab}) = \tau^{W,c} \cdot y^{lab}$. The income tax schedule is progressive and follows the following functional form (see e.g., Gouveia and Strauss, 1994 or García-Miralles et al., 2019)

$$T(y) = y\tau_0 \left[1 - (\tau_2 y^{\tau_1} + 1)^{-1/\tau_1}\right], \tag{186}$$

with average tax rate $T(y)/y$ and marginal tax rate $T'(y)$ given as

$$\tau^{W,i}(y) \equiv T(y)/y = \tau_0 \left[1 - (\tau_2 y^{\tau_1} + 1)^{-1/\tau_1}\right], \tag{187}$$

$$\tilde{\tau}^{W,i}(y) \equiv T'(y) = \tau_0 \left[1 - (\tau_2 y^{\tau_1} + 1)^{-1/\tau_1 - 1}\right], \tag{188}$$

where τ_1 and τ_2 govern the curvature of the tax function, and τ_0 is a scaling factor that leaves progressivity untouched. Note that for a good empirical fit, the input y of the tax function has to be normalized, i.e., divided by the average tax base $\bar{y}$. We can therefore write the overall deduction rates (average and marginal) as

$$\tau^{W,a,s} = \tau^{W,p} + \tau^{W,c} + \tau^{W,i}(\hat{y}^{lab,a,s}), \tag{189}$$

$$\tilde{\tau}^{W,a,s} = \tau^{W,p} + \tau^{W,c} + \tilde{\tau}^{W,i}(\hat{y}^{lab,a,s}), \tag{190}$$

$$\hat{y}^{lab,a,s} = (1 - \tau^{W,p} - \tau^{W,c})\frac{y^{lab,a,s}}{\bar{y}^{lab,a,s}}. \tag{191}$$

Given the principle of individual taxation, we assume that pensions are taxed independently of the respective representative household's labor income. The same functional form is assumed for the taxation of retirees. Typically, we will set $\tau^{P,p} = 0$. Note that for retirees, we do not require a marginal tax rate.

[42]This characterizes the Austrian tax system, with the important exception that the Austrian system has a lower and an upper threshold on contributions. However, in the current implementation, representative households typically fall within the region where contributions increase linearly in labor income.

3.5 Demand from Abroad

Export demand

The downward-sloping demand function for the domestic variety bundle that closes the model is assumed to be of the following form[43].

$$E^h = \lambda^E \ln(p^{E,h}/E^0), \tag{192}$$

where E^0 is a scaling factor, an indicator for world demand, and $\lambda^E < 0$ is the demand semi-elasticity parameter. For a 1% increase in the domestic export price index, exports drop by λ^E% of GDP (in calibration where GDP is normalized to 100).

Foreign demand for domestic assets

We assume demand functions responding to changes in domestic asset return factors of iso-elastic form, following Keuschnigg and Dietz (2007).

$$\hat{A}^{*i,h} = \hat{A}_0^{*i,h} \left(R^{i,h}\right)^{\lambda^{i,h}}, \quad i \in \{V, G\} \tag{193}$$

where $\hat{A}_0^{*i,h}$ is a scaling factor, and $\lambda^{i,h} > 0$ is the demand elasticity.

3.6 Production

The model allows for two types of production functions: one in which labor types might be imperfect substitutes to one another but all have the same elasticity of substitution w.r.t. capital (standard CES production function), and one that allows for different capital-skill substitutabilities (nested CES production function). Production is organized in several stages. The first stage is the same for both types of production functions and is presented first.

[43] An alternative assumption would be $E^h = E^0(p^{E,h})^{\lambda^E}$.

Age-dependent labor demand

In the first stage, young and old labor of a specific skill is combined into a labor composite of that skill L^s. The following functional form is assumed:

$$L^s = \begin{cases} \left[(\xi^{L^s})^{1-\varepsilon^{L^s}} \left(L^{Y,s}\right)^{\varepsilon^{L^s}} + (1-\xi^{L^s})^{1-\varepsilon^{L^s}} \left(L^{O,s}\right)^{\varepsilon^{L^s}} \right]^{1/\varepsilon^{L^s}}, & \text{if } \varepsilon^{L^s} \neq 0, \\ \left(\frac{L^{Y,s}}{\xi^{L^s}}\right)^{\xi^{L^s}} \left(\frac{L^{O,s}}{1-\xi^{L^s}}\right)^{1-\xi^{L^s}}, & \text{if } \varepsilon^{L^s} = 0. \end{cases}$$

Compensated unit demands $l^{Y,s}$ and $l^{O,s}$, and the unit expenditure function are computed by solving[44]

$$w^s = \min_{l^{Y,s}, l^{O,s}} \left\{ w^{Y,s} l^{Y,s} + w^{O,s} l^{O,s} \right\} \quad \text{s.t.} \quad L^s(l^{Y,s}, l^{O,s}) = 1. \tag{194}$$

As a solution, we get the usual unit expenditure function w^s and unit demand functions for the CES form.

$$w^s = \begin{cases} \left[\xi^{L^s} \left(w^{Y,s}\right)^{1-\lambda^{L^s}} + (1-\xi^{L^s}) \left(w^{O,s}\right)^{1-\lambda^{L^s}} \right]^{\frac{1}{1-\lambda^{L^s}}}, & \text{if } \varepsilon^{L^s} \neq 0, \\ w^s = (w^{Y,s})^{\xi^{L^s}} (w^{O,s})^{1-\xi^{L^s}}, & \text{if } \varepsilon^{L^s} = 0. \end{cases} \tag{195}$$

$$l^{Y,s} = \xi^{L^s} \left[w^s / w^{Y,s} \right]^{\lambda^{L^s}}, \qquad l^{O,s} = (1-\xi^{L^s}) \left[w^s / w^{O,s} \right]^{\lambda^{L^s}}, \tag{196}$$

where $\lambda^{L^s} = 1/(1-\varepsilon^{L^s})$ is the elasticity of substitution. The solution for splitting composite labor is then simply $L^{Y,s} = l^{Y,s} \cdot L^s$ and $L^{O,s} = l^{O,s} \cdot L^s$. Marginal products are $\partial L^s / \partial L^{Y,s} = \xi^{L^s} (L^s / L^{Y,s})^{1-\varepsilon^{L^s}}$ and $\partial L^s / \partial L^{O,s} = (1-\xi^{L^s})(L^s / L^{O,s})^{1-\varepsilon^{L^s}}$.

3.6.1 Standard CES production function

In the standard CES production function case, in the next stage, all skill composites are combined into a labor composite according to:

$$L = \begin{cases} \left[\sum_s (\alpha^{L,s})^{1-\varepsilon^L} (L^s)^{\varepsilon^L} \right]^{1/\varepsilon^L}, & \text{if } \varepsilon^L \neq 0, \\ \prod_s (L^s / \alpha^{L,s})^{\alpha^{L,s}}, & \text{if } \varepsilon^L = 0, \end{cases} \tag{197}$$

[44]Note that the unit demand functions are homogeneous of degree one. Hence, as long as payroll taxes τ^F are proportional and non-differentiated by type (Y vs. O), it does not matter whether they enter the cost minimization problem or are multiplied with w^s afterwards.

with $\sum_s \alpha^{L,s} = 1$. Marginal products are $\partial L/\partial L^s = \alpha^{L,s}(L/L^s)^{1-\varepsilon^L}$. In the last stage, the labor composite is combined with capital according to

$$Y = \begin{cases} \Phi \cdot \left[(\alpha^K)^{1-\varepsilon} K^{\epsilon} + (1-\alpha^K) L^{\varepsilon}\right]^{1/\varepsilon}, & \text{if } \varepsilon \neq 0, \\ \Phi \cdot (K/\alpha^K)^{\alpha^K} (L/(1-\alpha^K))^{1-\alpha^K}, & \text{if } \varepsilon = 0. \end{cases} \tag{198}$$

Marginal products are $\partial Y/\partial K = \Phi \cdot \alpha^K (Y/(\Phi K))^{1-\varepsilon}$ and $\partial Y/\partial L = \Phi \cdot (1-\alpha^K)(Y/(\Phi L))^{1-\varepsilon}$. The limiting case $\varepsilon \to 0$ gives a simple Cobb-Douglas production function.

3.6.2 Nested CES production function with capital-skill complementarity

This production function is of nested CES form to model different complementarity or substitutability characteristics between the input factors. The production function is specified to inherit capital-skill complementarity. In the implementation of the model, we use four input factors for the production function: $K, L^{D,1}, L^{D,2}, L^{D,3}$. Note that this does not restrict us to using three skill classes on the supply side, as several labor supply types can be aggregated into one of the three labor production inputs, but it obviously implies three equilibrium wage rates. In addition, we allow for skill-biased technological change governed by the potentially time-varying parameter ϱ that enters the production function together with capital as $\bar{K} = \varrho K$. The effect of a change in ϱ will be most pronounced for the labor type with the highest complementarity to capital. To save notation, we drop time indices and denote labor inputs simply as L_1, L_2, and L_3. The production function is given by the following form:

$$\begin{aligned} Y =& f(L_1, L_2, L_3, K) = \Phi \cdot z_1, \\ &\text{where} \\ z_1 =& \left[a_1 L_1^{\kappa_1} + (1-a_1) z_2^{\kappa_1}\right]^{\frac{1}{\kappa_1}} \\ z_2 =& \left[a_2 L_2^{\kappa_2} + (1-a_2) z_3^{\kappa_2}\right]^{\frac{1}{\kappa_2}} \\ z_3 =& \left[a_3 L_3^{\kappa_3} + (1-a_3) \bar{K}^{\kappa_3}\right]^{\frac{1}{\kappa_3}} \end{aligned} \tag{199}$$

Marginal products are given as

$$
\begin{aligned}
Y_{L_1} =& \Phi \cdot a_1 z_1^{1-\kappa_1} L_1^{\kappa_1 - 1} && (200)\\
Y_{L_2} =& \Phi \cdot (1-a_1) a_2 z_1^{1-\kappa_1} z_2^{\kappa_1-\kappa_2} L_2^{\kappa_2 - 1} && (201)\\
Y_{L_3} =& \Phi \cdot (1-a_1)(1-a_2) a_3 z_1^{1-\kappa_1} z_2^{\kappa_1-\kappa_2} z_3^{\kappa_2-\kappa_3} L_3^{\kappa_3 - 1} && (202)\\
Y_K =& \Phi \cdot (1-a_1)(1-a_2)(1-a_3) z_1^{1-\kappa_1} z_2^{\kappa_1-\kappa_2} z_3^{\kappa_2-\kappa_3} K^{\kappa_3 - 1} \varrho^{\kappa_3} && (203)
\end{aligned}
$$

The second derivative w.r.t. K is given as

$$
\begin{aligned}
Y_{KK} =& Y_K \Bigg[(\kappa_3 - 1)/K + (\kappa_2 - \kappa_3)/z_3 \frac{\partial z_3}{\partial K} \\
& + (\kappa_1 - \kappa_2)/z_2 \frac{\partial z_2}{\partial z_3} \frac{\partial z_3}{\partial K} + (1-\kappa_1)/z_1 \frac{\partial z_1}{\partial z_2} \frac{\partial z_2}{\partial z_3} \frac{\partial z_3}{\partial K} \Bigg], \qquad (204)
\end{aligned}
$$

with

$$
\begin{aligned}
\frac{\partial z_3}{\partial K} =& (1-a_3) \left(\frac{z_3}{K} \right)^{1-\kappa_3} \varrho^{\kappa_3}, \\
\frac{\partial z_2}{\partial z_3} =& (1-a_2) \left(\frac{z_2}{z_3} \right)^{1-\kappa_2}, \\
\frac{\partial z_1}{\partial z_2} =& (1-a_1) \left(\frac{z_1}{z_2} \right)^{1-\kappa_1}.
\end{aligned}
$$

Let us define income shares as

$$
\alpha_i = Y_{L_i} L_i / Y, \quad i \in \{1,2,3\} \quad \text{and} \quad \alpha_K = Y_K K / Y, \qquad (205)
$$

and recall that $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_K = 1$ because of linear homogeneity. For given income shares and elasticity parameters κ_i, we can compute the corresponding share parameters a_i in the process of calibration.

$$
\begin{aligned}
\frac{\alpha_3}{\alpha_K} =& \frac{a_3}{1-a_3} \left(\frac{L_3}{\bar{K}} \right)^{\kappa_3} \quad \Rightarrow \quad a_3 = \frac{1}{1 + \frac{\alpha_K}{\alpha_3} \left(\frac{L_3}{\bar{K}} \right)^{\kappa_3}} && (206)\\
\frac{\alpha_2}{\alpha_3 + \alpha_K} =& \frac{a_2}{1-a_2} \left(\frac{L_2}{L_3} \right)^{\kappa_2} \quad \Rightarrow \quad a_2 = \frac{1}{1 + \frac{\alpha_3 + \alpha_K}{\alpha_2} \left(\frac{L_2}{L_3} \right)^{\kappa_2}} && (207)\\
\frac{\alpha_1}{\alpha_2 + \alpha_3 + \alpha_K} =& \frac{a_1}{1-a_1} \left(\frac{L_1}{L_2} \right)^{\kappa_1} \quad \Rightarrow \quad a_1 = \frac{1}{1 + \frac{\alpha_2 + \alpha_3 + \alpha_K}{\alpha_1} \left(\frac{L_1}{L_2} \right)^{\kappa_1}} && (208)
\end{aligned}
$$

These equations have been computed sequentially by first dividing (202) by (203). Then, use (205) to get expression (206). For (207), divide (202) by (201), use (205), the definition of z_2, and (206), and so forth.

The notion of capital-skill complementarity is characterized by a parameterization of the production function such that the following ordering of Allen-Uzawa partial elasticities of substitution holds:

$$\sigma^{K,L_1} > \sigma^{K,L_2} > \sigma^{K,L_3}. \tag{209}$$

The Allen-Uzawa partial elasticities of substitution measure the percentage change in the ratio of two inputs in response to a change in the ratio of the corresponding input prices, holding all other input prices and the output quantity (but not the other inputs) constant. Jaag (2005) provides a detailed discussion and derivation of the mapping of σ^{K,L_i} into the calibration parameters $\kappa_i, \forall i \in \{1,2,3\}$. In summary, the mapping is given by

$$\sigma_1 = \sigma^{L_1,K}, \tag{210}$$

$$\sigma_2 = (1-\alpha_1)\left(\sigma^{L_2,K} - \sigma_1\right) + \sigma_1, \tag{211}$$

$$\sigma_3 = (1-\alpha_1-\alpha_2)\left(\sigma^{L_3,K} - \frac{\sigma_2}{1-\alpha_1}\right) + \sigma_2, \tag{212}$$

where $\sigma_i = 1/(1-\kappa_i)$. Note that the calibration of κ_i does not depend on any other structural parameters in the production function and can, therefore, be done as a first step before fixing a_i using equations (206) to (208).

3.6.3 Public capital stock total factor productivity link

For the productivity index Φ, we assume a simple iso-elastic form[45].

$$\Phi(K^G) = A^0 \cdot \left(K^G\right)^{\sigma^G}, \tag{213}$$

where σ^G is interpreted as the elasticity of output w.r.t. public capital.

[45] A^0 denotes a scaling factor and is not to be confused with assets at time t: A_t.

4 Solution Algorithm

The model is solved from 1 to $\bar{t}$[46] using a "Hybrid Tatonnement" algorithm, which proves very speed efficient in comparison to classical tatonnement methods or Fair and Taylor (1983)-type algorithms that are often used for solving perfect foresight models. The algorithm is labeled 'hybrid' because it combines Newton's method[47] applied to a subset of the system of equilibrium conditions stacked for all points in time with classical tatonnement. Newton's method is used to solve the two-point boundary problem of capital accumulation for given prices. Insert (106) to eliminate p_t^K in (125). Then (122), (124), and (125) form a system of three equations and three unknowns (K_t, q_t, I_t) in t. Stacking the conditions over time implies that in each iteration step of Newton's method, the Jacobian matrix of size $3*(\bar{t}-1) \times 3*(\bar{t}-1)$ has to be inverted. This can be done very efficiently using sparse matrices and providing the analytic solutions of the derivatives.[48] However, this approach does not scale well to solving the household side if households are heterogeneous.[49] Instead, we exploit finite lifetimes and the specific structure of the dynamic household problem and solve it individually by cohort, making use of parallelization using multiple CPU cores. An individual household's problem is solved as follows: For given prices, taxes, etc., we make a guess for the shadow price of assets for the first active age $\lambda_{\underline{z}}^a$, which is then solved forward in age using (37). All labor supply decisions, pension point accumulations, updating of average and marginal tax rates, etc., are solved by simple block iteration, where we have to make case distinctions as the policy functions can be discontinuous.[50] We then use the secant method to find $\lambda_{\underline{z}}^a$ that will set savings at maximum age to 0. Once optimal firm and household behavior is computed for given vectors of prices, we use a standard tatonnement method to update the price vectors. A sketch of the algorithm follows:

[46]Period 0 denotes the initial steady state.

[47]This is how perfect foresight models are solved in DYNARE; see Juillard (2018).

[48]Using an optimized sparse matrix solver, this typically takes less than 20 ms for 500 periods on a regular PC.

[49]For example, in our case, with 86 (economically active) age groups, 3 skill levels, 10 control and state variables, and simulating 500 periods, the Jacobian matrix would be of dimension $1\,290\,000 \times 1\,290\,000$. Extending heterogeneity far beyond 3 skill levels would therefore prove unfeasible, while perfectly manageable using the "Hybrid Tatonnement" method.

[50]This can occur when a small change, e.g., in the wage rate implies that the current income starts (or stops) to be part of the pension base, which will lead to a jump in the effective tax rate and therefore labor supply.

The algorithm steps[51]:

1. Make an initial guess for labor demand $\left\{L_t^{D,s}\right\}_{t=1}^{\bar{t}}$ and prices $\left\{p_t^h\right\}_{t=1}^{\bar{t}}$, $\left\{r_t^{V,h}\right\}_{t=1}^{\bar{t}}$, etc.

2. Solve the two-point boundary value problem of capital accumulation by Newton's method based on the initial state K_1, labor demands and prices for all t to get a new series of $\{K_t\}_{t=1}^{\bar{t}}$, $\{q_t\}_{t=1}^{\bar{t}}$, $\{I_t\}_{t=1}^{\bar{t}}$. Compute $\{Y_t\}_{t=1}^{\bar{t}}$, $\{w_t^s\}_{t=1}^{\bar{t}}$, $\{\theta_t^s\}_{t=1}^{\bar{t}}$ and $\{V_t\}_{t=1}^{\bar{t}}$, etc.

3. Solve the household problems for all cohorts for given wage rates (in parallel) and aggregate over time to compute $\left\{L_t^{S,s}\right\}_{t=1}^{\bar{t}}$, $\{C_t\}_{t=1}^{\bar{t}}$, $\{A_t\}_{t=1}^{\bar{t}}$, etc.

4. Check Walras' Law.

5. Rescale the asset position in the shock period of all households such that $A_1 = V_1 + D_1^F + D_1^G$.

6. Compute $D_t^F = A_t - V_t - D_t^G$ for $t = 2, \ldots, \bar{t}$ residually and use that to compute the trade balance and exports.

7. Rescale the profiles for accidental bequest received to match aggregate accidental bequests given for all periods.

8. Rescale a government budget instrument of choice in order to fulfill given a budget rule, i.e., given path of $\left\{D_t^{G*}\right\}_{t=1}^{\bar{t}}$.

9. Impose labor market clearing to update labor demand ($L_t^{D,s} = L_t^{S,s}, \ \forall t$).

10. Update $\left\{p_t^h\right\}_{t=1}^{\bar{t}}$ to clear the domestic goods market, if λ^E is finite.

11. Update $\left\{r_t^{i,h}\right\}_{t=1}^{\bar{t}}$, $i \in \{V, G\}$ to clear the domestic asset markets, if $\lambda^{i,h}$ is finite.

12. Repeat steps 2-11 until prices converge for all t.

Unanticipated shocks are introduced by first solving the model from 1 to $\bar{t}$ without the shock. Then, a shock is introduced, and the model is solved again, but only from the shock period $t^s > 1$ to $\bar{t}$. The shock period t^s is the period when agents

[51] K_1, D_1^F and D_1^G are predetermined and equal to their initial steady state values from $t = 0$.

learn about the shock, and not necessarily when, e.g., a new policy is introduced.

The model is implemented using the programming language R. The numerically demanding part of computing transition paths with the algorithm above is carried out in C++ using the linear algebra library Armadillo (Sanderson and Curtin, 2016 and Eddelbuettel and Sanderson, 2014), the sparse matrix solver library SuperLU, and the parallel programming API OpenMP.[52] Codes for solving a plain Auerbach-Kotlikoff model using the same algorithm are available here: `https://github.com/solveCGE/solveOLG_doc`. Table 4.1 shows that rewriting tight loops in C++ and solving the household problem in parallel can dramatically reduce runtimes compared to a single-threaded, pure R implementation.

Table 4.1: Runtimes of computing the baseline (over 1'000 periods) as featured in Fiscal Advisory Council (2025)

language	R only	R/C++	R/C++	R/C++
threads	1	1	8	40
hours	9	0	0	0
minutes	41	3	0	0
seconds	42	41	57	20

[52] See `https://www.r-project.org`, `https://gcc.gnu.org`, `http://arma.sourceforge.net`, `https://cran.r-project.org/web/packages/RcppArmadillo/`, `https://portal.nersc.gov/project/sparse/superlu/`, and `https://www.openmp.org/`.

5 Matching the Data

In principle, one can differentiate two calibration strategies that can be used to fit deterministic, dynamic general equilibrium models to the data. The more conventional approach is to fix a base year, typically the last year for which sufficient data are available, and treat it as the initial steady state of the model. Matching the steady state of the model to the data of the base year is – in principle – a rather trivial problem. However, the fundamental problem with this approach is that, in reality, the economy is typically far from stationary in the base year. For example, with ongoing population aging, the current age structure of the population is considerably younger than the stationary age structure given current vital rates. Another limitation of the 'base year steady state' approach is that one cannot replicate the coexistence of a negative primary balance and a positive debt level in the base year, or a positive current account and positive net foreign assets, to provide just two examples. One is further limited in adequately capturing ongoing transformation processes, such as structural reforms that have been implemented in the past but are lasting into/transforming the future (e.g., fundamental pension reforms). To cope with these problems, the model – in its latest version – allows for a different calibration strategy; one where the steady state is placed way back in time, and the model is dynamically fitted to replicate the evolution up to the chosen base year. Naturally, this approach is considerably more sophisticated. First, one has to partition the parameter set into time-dependent and time-independent parameters. The former set mainly includes policy parameters, such as tax rates, scaling factors for age- and skill-dependent average cost profiles for public consumption and transfers, which are set to replicate the historical revenues and expenditures.[53] An exception to this procedure of fitting fiscal variables is pension expenditure, which is simply the result of changes in the pension parameters over time (see Section 5.4). We also treat (historical) productivity growth g_t as time-dependent, i.e., it reflects the unexplained residual that does not result from changes in the production factors. To uphold the predictive power of the model, the set of time-invariant parameters was kept as large as possible.[54] The set of time-invariant parameters typically includes

[53] As tax bases are endogenous and therefore 'moving targets', this type of fitting is done on-the-fly during a simulation run, i.e., as an extra step 11b in the solution algorithm explained in the previous section.

[54] For the time after the base year (out of sample), we keep all time-variant parameters constant except for already known changes, e.g., enacted future tax cuts.

deep parameters governing agents' behavior[55] and technology. Of those, some are chosen based on suggestions in the literature (e.g., labor supply elasticities), while others are 'estimated' in order to give the best fit (e.g., capital-labor elasticity or capital share in production). The estimation is done by carrying out a grid search subject to minimizing the Mahalanobis distance error for a select group of time series (e.g., capital stock to GDP, consumption to GDP). The chosen/estimated parameter values naturally vary by model specification and are therefore – to some degree – application-specific. Section C in the appendix lists the parameter values by application. The fit of the historical data is displayed in Figures 5.1 to 5.3 for the latest application (Fiscal Sustainability Report 2021). The remainder of this section zeros in on the data and assumptions that were used when fitting the model.

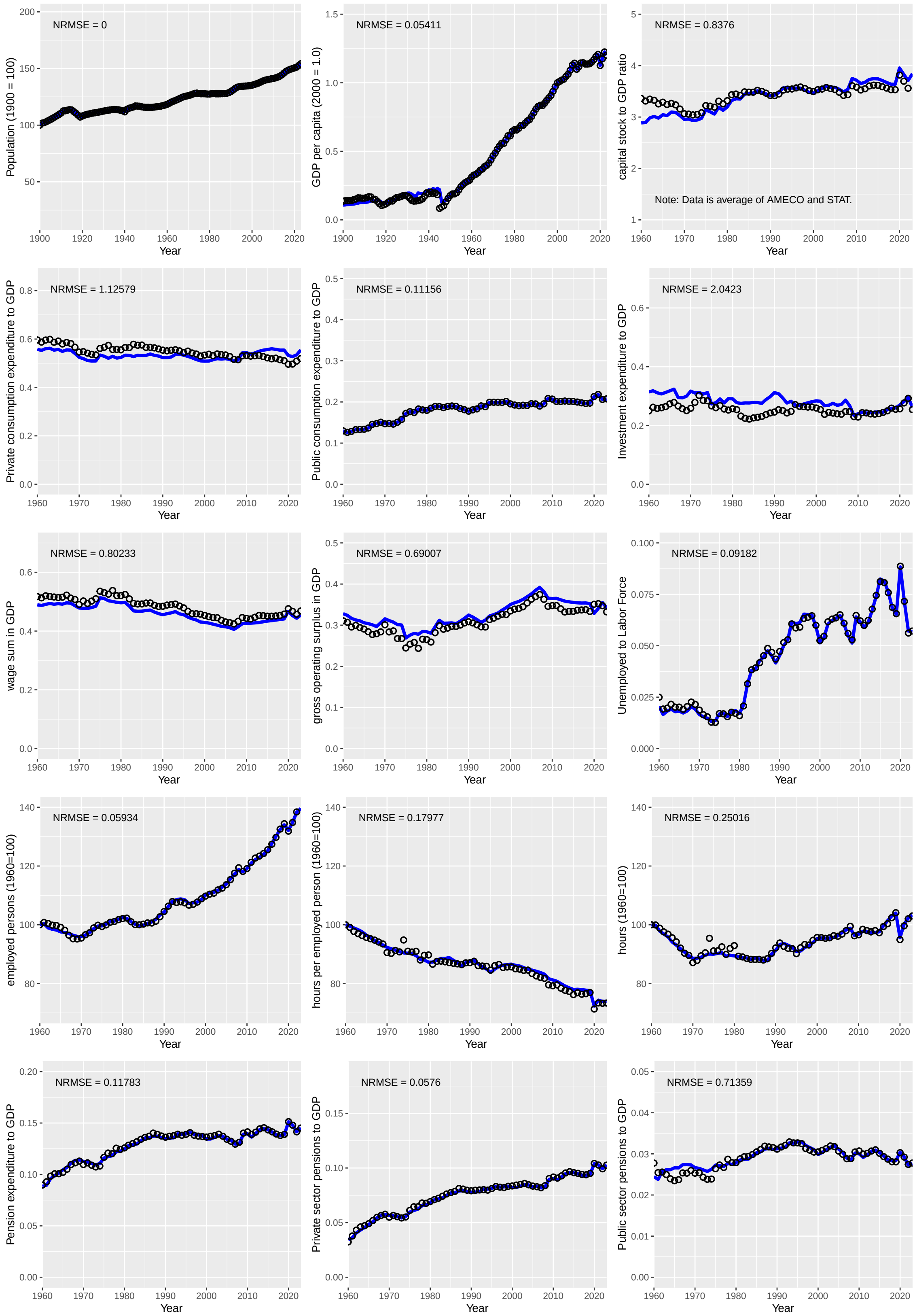
5.1 Demography and Education

The model takes demographic vital rates, i.e., fertility, mortality, and net migration, as exogenous inputs and computes the historical change in population size as well as future developments based on a given forecast.[56] By default, the model uses the base scenario of the latest population forecast of Statistics Austria (STAT) but may include its sensitivity scenarios, as well as Eurostat's population forecasts. Information on the historical development of vital rates and population was taken from the historical population reconstruction that goes back to 1650, used in Sánchez-Romero et al. (2024). The evolution of the skill composition of the population is exogenously fed into the model. In the current parameterization, three educational groups are represented in the model: low ($s = 1$), medium ($s = 2$), and high ($s = 3$). They are defined based on the highest educational attainment: low: compulsory education ('Pflichtschule'), medium: apprenticeships, intermediate technical and vocational schools, academic secondary schools, and higher technical and vocational schools ('Lehre', 'berufsbildende mittlere Schule (BMS)' and 'allgemeinbildende/berufsbildende höhere Schule (AHS/BHS)'), and high: tertiary, i.e., college and university education ('Universität und Fachhochschule'). The historical evolution of the educational distribution is derived from data by the Wittgenstein Centre (Lutz et al., 2018), while the future evolution is based on the simulation results

[55] One exception is the diverging historical trends of participation rates and hours per worker that cannot be rationalized in the given model set-up. We therefore impose trends in the corresponding labor supply scaling parameters: φ_0^{δ} and φ_0^{ℓ}.

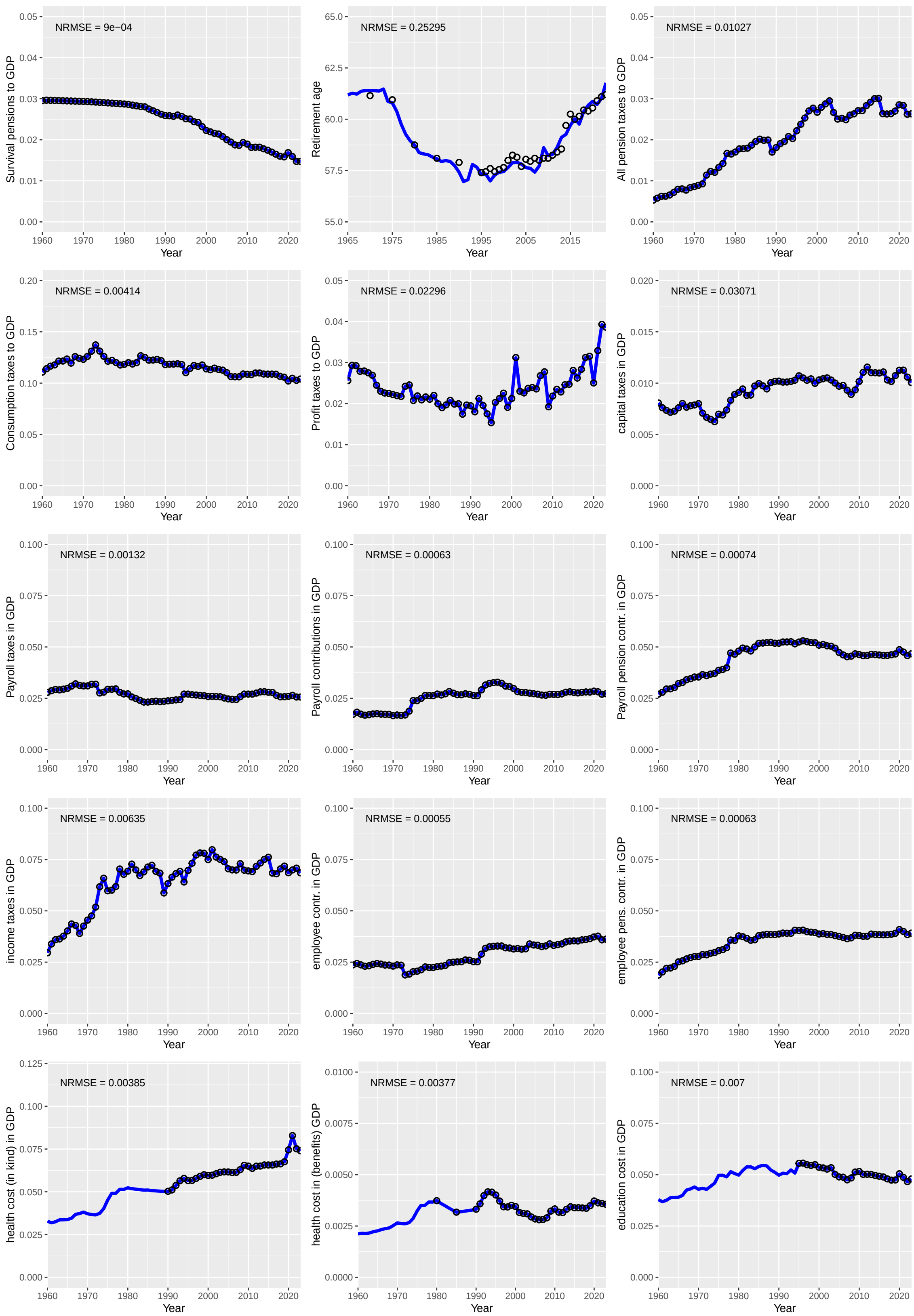
[56] Alternatively, if one wants to exactly replicate age-specific population sizes, one can fix cohort sizes, fertility, and mortality rates and set net migration to fulfill the demographic law of motions.

Figure 5.1: Historical data fit 1 of 3



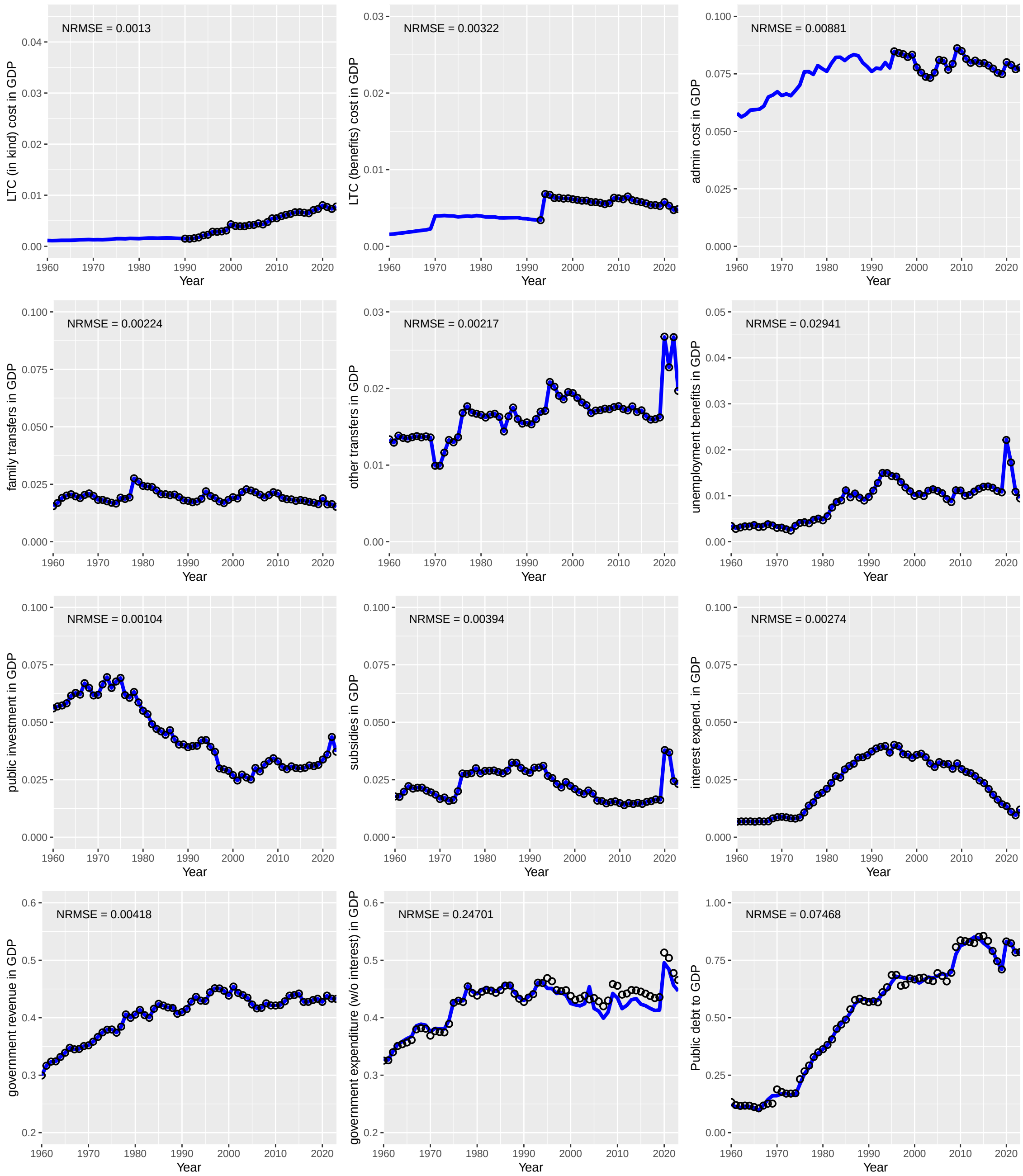
Note: Solid line shows model outcome, dots are data (some historically linked, and backcasted).

Figure 5.2: Historical data 2 of 3



Note: Solid line shows model outcome, dots are data (some historically linked, and backcasted).

Figure 5.3: Historical data 3 of 3



Note: Solid line shows model outcome, dots are data (some historically linked, and backcasted).

in Sánchez-Romero et al. (2024). In order to capture the differences in vital rates by education, we have adjusted the population forecasts, which are typically not education-specific. Information on dispersion in mortality is taken from education-specific life tables for Austria from Klotz and Asamer (2014) and is used to adjust the model's education-specific mortality rates under the constraint of keeping the total number of deaths in a given year unchanged.

5.2 Macro Aggregates

The model is fit to the Austrian national accounts data (as provided by STAT) according to the European System of Accounts (ESA 2010) and previous national accounts standards (ESA 1995, SNA 1968), linked accordingly. Following the production approach, gross domestic product (GDP) is gross value added plus taxes (less subsidies) levied on products (D.21 – D.31). GDP by the income approach is wage-related income, capital-related income, and product and other production taxes (D.2) minus all subsidies (D.3).[57] GDP by the expenditure approach is expenditure for private consumption, public consumption, investment (private and public), and net exports.

$$\begin{aligned}
\text{(production) } GDP_t &= p_t^h(Y_t - \bar{f}_t - cV_t) + T_t^C, && (214)\\
\text{(income) } GDP_t &= (1+\tau_t^F)w_t L_t + (1+\tau_t^K)p_t^K K_t + T_t^C + T_t^Y - Sub_t^f, && (215)\\
\text{(expenditure) } GDP_t &= p_t^C C_t + p_t^{C^G} C_t^G + p_t^I I_t + p_t^{I^G} I_t^G + p_t^{E,h} E_t^h\\
&\quad - \tilde{p}_t^m \sum_z z_t^m, && (216)
\end{aligned}$$

for $z \in \{C, C^G, I, I^G\}$. One can compute (detrended) GDP at constant prices by simply using prices of a reference year, e.g., $t = 0$, in the definitions above.

$$\begin{aligned}
\text{(production)} \quad GDP_t &= p_0^h(Y_t - \bar{f}_t - cV_t) + \tau_0^C \tilde{p}_0^C C_t + \tau_0^{C^G} \tilde{p}_0^{C^G} C_t^G && (217)\\
\text{(expenditure)} \quad GDP_t &= p_0^C C_t + p_0^{C^G} C_t^G + p_0^I I_t + p_0^{I^G} I_t^G + p_0^{E,h} E_t^h\\
&\quad - \tilde{p}_0^m \sum_z z_t^m, && (218)
\end{aligned}$$

The model does not distinguish between employees and self-employed persons. As the taxation of self-employed persons is more similar to that of employees than to

[57]Note that Sub^I and Sub^l cancel out in this expression. They are part of capital-related income $(1+\tau_t^K)p_t^K K_t + Sub_t^I + Sub_t^l$ and part of total subsidies $Sub_t^I + Sub_t^l + Sub_t^f$.

corporate firms, self-employed persons are treated as employees. This implies that the labor share $1 - \alpha_k$ has to be adjusted accordingly, as self-employed incomes in ESA are part of gross operating surpluses (B.2g) instead of compensation of employees (D.1). A detailed description of the matched data is provided in the "Historical Macro and Fiscal Database for Austria" (HMFDA).[58]

5.3 Household Data

In order to align age- and skill-specific household profiles with their empirical counterparts, several microdata sets have been used. First, the "register-based labor market statistics" (RBLMS) ("Abgestimmte Erwerbstatistik") were used to compute age- and skill-specific participation and unemployment rates. We further derive job-flow transitions and family-status transitions from the RBLMS. For the historical development of age- and skill-specific profiles, we used the "Microcensus," which is less detailed but than the RBLMS but provides longer time series. Third, the EU statistics on income and living conditions (EU-SILC) were used to retrieve income (e.g., work income, transfers) and hours profiles across age and skills. By default, labor tax rates are not derived from SILC data but are computed endogenously by using estimated tax functions. Data for these estimations stem from the historical tax-liability simulator (TaLiS) by Reiss and Schuster (2020) and historical wage tax statistics by STAT. Fourth, further life-cycle information was taken from the "National Transfer Accounts" (NTA) for Austria (see United Nations, 2013, for methodological background). Hammer (2015) computes various age-dependent per capita profiles differentiated into three skill groups that were directly used in the model. These profiles include average cost profiles for public health care, long-term care (LTC), and education expenditure. We allow for shifting average cost profiles for health care and LTC to capture changes in health conditions.

5.4 Fiscal Data and the Austrian Pension System

Fiscal data are based on the national accounts government account. However, not all parts of the government sector according to ESA are taken into account in the model (see Table 5.1). Most importantly, firms in the government sector are treated as private firms. On the government revenue side, this implies no revenue from

[58]The database was constructed for matching the model to historical macroeconomic and fiscal data and is available at `https://www.fiskalrat.at` (see Schuster, 2025).

output (P.1O), which is part of government consumption (P.3) on the expenditure side, and property income (D.4). Further, there are no other current transfers (D.7) nor other capital transfers (D.99) received. Taxes (D.2 + D.5 + D.91) are regrouped by functions: labor, capital, interest, profits, pensions, consumption, and output. Based on the individual taxes according to the detailed national tax lists, all taxes are sorted into one of these functions. On the expenditure side, four types of expenditures are modeled: transfers to households, transfers to firms, public consumption and investment, and interest payments. Public consumption (≈ D.1 + D.29 + D.632 + parts of D.7[59]) is broken down by function into expenditure on health care, long-term care, and education, i.e., costs that in the model are explained by the age and skill structure of the population, and a residual term (broadly administrative costs). For the break-down by function, we used COFOG ("Classification of the Functions of Government"), ESSPROS ("European System of Integrated Social Protection Statistics"), and SHA ("System of Health Accounts") data. To approximately match the historical development of public debt, historical stock-flow adjustments were smoothed and treated as financial transactions sent abroad, which are part of the primary balance. For an exact replication of public debt and the primary balance, which can be of relevance for recent years when relating simulation results to EU fiscal rules nuances, there is the option of explicitly taking stock-flow adjustments into account (see Section 5.5). More information on the fiscal data items is provided in the HMDFA.

The Austrian pension system

A special focus was put on modeling the Austrian pension system and capturing the ongoing effects of historical reforms. In the recent past, the most important reforms were introduced in the early 2000s. Next to harmonizing the various pension systems, these reforms brought a step-wise extension of the reference period for calculating pension benefits toward lifetime income, becoming the assessment basis (starting from only considering the best 15 income years for regular employees ("ASVG") and the last income year for civil servants, respectively). To capture these reforms, the model by default distinguishes between three different systems: $\mathcal{P} = \{A, B, N\}$; the old (prior to the reforms in the early 2000s) system for non-civil servants (A, "Altrecht"), the old system for civil servants (B, "Beamte"), and

[59] We include public transfers to privately run education and health facilities, recorded in D.7, in public consumption.

Table 5.1: Representation of the ESA 2010 government account in the model

Revenue	Expenditure
P.1O Total sales ✓ (⇒)	P.2 Intermediate consumption ✓
D.2 Indirect taxes ✓	D.1 Compensation of employees ✓
D.4 Property income (rec.) ✗	D.29 Other taxes on production (pay.) ✓
D.5 Direct taxes ✓	D.3 Subsidies ✓
D.6 Social contributions ✓	D.41 Interest (pay.) ✓
D.7 Other current transfers (rec.) ✗	D.62 Social benefits in cash ✓
D.91 Asset taxes ✓	D.632 Social transfers in kind ✓
D.99 Other capital transfers (rec.) ✗	D.7 Other current transfers (pay.) ✗ and ✓
Other ✗	D.92 Investment grants ✓
	D.99 Other capital transfers (pay.) ✗
	P.5 Gross capital formation ✓
	Other ✗

Note: ✓ explicitly modeled, ✗ not explicitly modeled, ⇒ part of the computation of government consumption.

the new uniform system (N, "Pensionskonto"). The systems are parameterized to reflect the differences in pensionable years, accrual rates, indexation rules, and replacement rates. The parameterization for currently retiring cohorts is shown in Table 5.2. The full collection of pension parameters for different birth cohorts is available in the HMFDA. In the transition phase, $\alpha_z^{A,a}$ and $\alpha_z^{B,a}$ are linearly faded to 0, which reflects that persons are entitled to pension benefits according to the different systems, weighted by the relative duration of contribution ("pro rata temporis"). The historical evolution of the "Altrecht" system parameterization is frozen after 2003 (we assume a binding loss cap, i.e., "Verlustdeckel"). The model does not incorporate the mandatory transfer of mixed system persons into the new system with a one-time compensatory pension point credit ("Kontoerstgutschrift") in 2014, which makes little quantitative difference. The income cap is set such that about 90% of the wage sum falls below the threshold (based on wage tax statistics by STAT). Note that there is no income threshold in the old system for civil servants ("Beamte"), i.e., $y^{B,cap} = \infty$.

5.5 Wedges and Other Corrections

When matching the data, it is often required to replicate certain observed peculiarities that the model cannot do in its plain form. Therefore, in the implementation, many different wedges are added that work like implicit taxes or subsidies (i.e.,

Table 5.2: Current parameterization of Austrian pension systems (for persons retiring in 2024)

	"Altrecht" $p = A$	"Beamte" $p = B$	"Pensionskonto" $p = N$
$G_t^{p,a}$ $(a < a^R)$	1	1	$\mathcal{G}_t$
$G_t^{p,a}$ $(a \geq a^R)$	1	1	1
$m_z^{p,a}$	$0.8/40 = 0.0200$	$1.0/45 = 0.0222$	$0.8/45 = 0.0178$
$\varsigma_z^{GW,p,a} - 1$ (males)	-4.2%/+4.2% $\times\lvert a - a^R\rvert$	-3.36%/+0.0% $\times\lvert a - a^R\rvert$	-5.1%/+5.1% $\times\lvert a - a^R\rvert$
$\varsigma_z^{CY,p,a}$	$\max(1, \frac{\text{contrib. years}}{36})$	$\max(1, \frac{\text{contrib. years}}{32})$	1

Note: 36 and 32 refer to the number of best years according to "Altrecht" that are relevant in the year 2024. Gruber-Wise penalties/rewards (ς^{GW}) just reported for males. These figures have to be averaged accordingly for unisex households.

not collected or paid by the government) to (at least temporarily) distort behavior. This can be an additional distortionary tax that is reimbursed in a lump-sum fashion such that resources are unaltered, or by directly adding a wedge to a first-order condition. An example would be to introduce a consumption wedge during the COVID-19 pandemic to replicate the strong drop in consumption due to lockdowns. The model allows for similar wedges at all main decision margins. Wedges can also work as convenient interfaces to other smaller models more focused on a specific aspect (see section 6.1). They prove to be a useful concept to incorporate specific channels without doing a full-fledged model extension inside an already large model.

Another type of correction is implicit side-payments that, in contrast to wedges, do not alter the first-order conditions but the resource constraints. Again, this can be useful to bring the model closer to the data without substantially increasing the model's complexity. An example is implicit side payments between employees and employers to match fluctuations and deviations from the long-run trend in the labor share. This way, one can capture the effects of more elaborate wage negotiation mechanisms, and it is much more robust than trying to work with time-varying elasticity parameters in the production function.[60]

Some applications might require explicitly accounting for stock-flow adjustments (SFA) to public debt. This is done by defining D^G as completely absent of stock-

[60] Getting the historic development of the labor share right is, for example, an important prerequisite for correctly computing today's pension entitlements.

flow adjustments and keeping track of stock-flow adjustments separately. As a simplification, we assume that stock-flow adjustments are a pure statistical correction and that they have no economic meaning in the model, i.e., they are assets that are not held by an economic agent and pay no interest. We define the stock of stock-flow adjustments as D^{SFA}, such that public debt including stock-flow adjustments is $D_t^{\overline{G}} = D_t^G + D_t^{SFA}$. Then, the stock D_t^{SFA} evolves as follows:

$$\hat{\mathcal{G}}_t \mathcal{G}_t^{€} D_{t+1}^{SFA} = D_t^{SFA} + SFA_t. \tag{219}$$

Taken together, the law of motion for (detrended) public debt, including stock-flow adjustments, is

$$\hat{\mathcal{G}}_t \mathcal{G}_t^{€} D_{t+1}^{\overline{G}} = D_t^{\overline{G}} - PB_t + Exp_t^R + SFA_t, \tag{220}$$

with $Exp_t^R = i_t^{G,h} \left[D_t^G - PB_t \right]$. Expressing debt levels as end-of-period using $D_t^{\overline{G},end} = \hat{\mathcal{G}}_t \mathcal{G}_t^{€} D_{t+1}^{\overline{G}}$ gives

$$D_t^{\overline{G},end} = D_{t-1}^{\overline{G},end} / (\hat{\mathcal{G}}_{t-1} \mathcal{G}_{t-1}^{€}) - PB_t + Exp_t^R + SFA_t, \tag{221}$$

5.6 Indicators

There are different indicators that can be used to condense the results of long-run fiscal sustainability analyses. We briefly describe three indicators and how they can be computed from the model. All three have in common that we compute required primary balance adjustments to comply with a specific public debt-to-GDP trajectory, but without including the actual feedback effects on the economy from implementing these adjustments. We start with the S1 and S2 indicators used by the European Commission (European Commission, 2024), and then derive the fiscal gap/fiscal space concept used in the Fiscal Sustainability Reports of the Austrian Fiscal Advisory Council.

S1 Indicator

The S1 indicator measures the permanent and constant adjustment of the primary balance (as a share of GDP) that is required starting $\underline{t}$ to let $D_t^{\overline{G},end}/GDP_t$ hit a specified target value in $\overline{t}$. The European Commission sets the target value to 0.6, and $\overline{t} = 2070$. One simply has to expand PB_t in (221) to $PB_t + S1 \cdot GDP_t$, find S1, and iterate forward from $\underline{t}$ such that the target is met in $\overline{t}$. Note that here, we assume the path of $i_t^{G,h}$ is unaffected by S1, while Exp_t^R is not.

S2 Indicator

In contrast to the S1 indicator, the S2 indicator does not use a terminal year. It is the permanent and constant adjustment of the primary balance (as a share of GDP) such that, if iterated forward indefinitely, the law of motion for public debt converges to a finite value. Because of the saddle-path stability, there is a single value for S2 for this to be true. For simplicity, we ignore future stock-flow adjustments for this exercise. Let us define $Q_t \equiv R_t^{G,h}/\hat{\mathcal{G}}_t$ as the effective interest factor, and the compound interest factor from period i to period j as $\alpha_{i,j} \equiv Q_i \times Q_{i+1} \times \cdots \times Q_{j-1} \times Q_j$, such that $D_{t+1}^G = Q_t \left[D_t^G - PB_t - S2 \cdot GDP_t\right]$.

Lemma 5.1. *The* ***S2 indicator*** *at $\underline{t}$ is given as*

$$S2 = \frac{D_{\underline{t}} - \sum_{i=\underline{t}}^{\infty} \frac{PB_i}{\alpha_{1,i-1}}}{\sum_{i=\underline{t}}^{\infty} \frac{GDP_i}{\alpha_{1,i-1}}} \tag{222}$$

The ***S2 indicator in finite time approximation*** *(with constant values marked with bars after $\overline{t}$) is given as*

$$S2 = \frac{D_{\underline{t}} - \left[\sum_{i=\underline{t}}^{\overline{t}} \frac{PB_i}{\alpha_{1,i-1}} + \frac{\overline{PB}}{\overline{q} \cdot \alpha_{1,\overline{t}-1}}\right]}{\left[\sum_{i=\underline{t}}^{\overline{t}} \frac{GDP_i}{\alpha_{1,i-1}} + \frac{\overline{GDP}}{\overline{q} \cdot \alpha_{1,\overline{t}-1}}\right]}, \tag{223}$$

with $\overline{q} = \overline{Q} - 1$.

The derivation steps are documented in Section A.

Fiscal Gap and Fiscal Space

We start by dividing (221) by GDP_t. Note that $G_t^{GDP} = 1 + g_t^{GDP}$ is the growth factor of nominal (undetrended) GDP from t to $t+1$, i.e., $G\tilde{D}P_{t+1} = (1 + g_t^{GDP})G\tilde{D}P_t$, or equivalently, $\hat{\mathcal{G}}_{t-1}\mathcal{G}_{t-1}^{€}GDP_{t+1} = (1 + g_t^{GDP})GDP_t$. We use the following notation to indicate GDP ratios: $\breve{X}_t \equiv X_t/GDP_t$.

$$\breve{D}_t^{\overline{G},end} = \breve{D}_{t-1}^{\overline{G},end}/(1 + g_{t-1}^{GDP}) - \breve{PB}_t + \breve{Exp}_t^R + \breve{SFA}_t. \tag{224}$$

Using the definition of the change in the debt-to-GDP ratio $\Delta\breve{D}_t^{\overline{G},end} = \breve{D}_t^{\overline{G},end} - \breve{D}_{t-1}^{\overline{G},end}$, we can rearrange to

$$\Delta\breve{D}_t^{\overline{G},end} = -Denom_t - \breve{PB}_t + \breve{Exp}_t^R + \breve{SFA}_t, \tag{225}$$

where $Denom_t = \breve{D}_{t-1}^{\overline{G},end} g_{t-1}^{GDP} / G_{t-1}^{GDP}$ is the GDP-denominator effect. Next, we split the primary balance, as described in Section 2.12, into $\breve{PB}_t = \breve{PB}_t^{npc} + \breve{PB}_t^{adjust}$, and rearrange again.

$$\underbrace{\breve{PB}_t^{adjust}}_{\text{“fiscal gap”}} = -Denom_t - \breve{PB}_t^{npc} + \breve{Exp}_t^R + \breve{SFA}_t - \Delta\breve{D}_t^{\overline{G},end}, \tag{226}$$

Where the left-hand side is the “fiscal gap,” i.e., the required adjustment in the primary balance at time t to comply with a targeted change in the debt-to-GDP ratio. If we flip signs, then we can define the “fiscal space” in t as $Denom_t + \breve{PB}_t^{npc} - \breve{Exp}_t^R - \breve{SFA}_t + \Delta\breve{D}_t^{\overline{G},end}$.

6 Extensions

6.1 Climate Module

The goal of the climate module was to introduce the budgetary effects of climate change and climate and energy policy to the model. The module was developed for the Fiscal Sustainability Report 2025 (Fiscal Advisory Council, 2025) and can optionally be soft-linked to the core OLG model. At its core, the module endogenizes energy consumption in production in a small separate model, with its results fed into the large OLG model. In addition, the module features channels that mostly rely on external analysis, which simply feed into the model. The rest of this section focuses on how these linkages work and the altered firm problem with energy as an additional input factor. For information on corresponding data, the reader is referred to section C.3 and Fiscal Advisory Council (2025).

6.1.1 Endogenous Energy Consumption

We formulate this as a firm problem[61] in a competitive environment, taking labor supply and the final good's price as given. The problem is split into different layers. The final good producer sources labor input L and a capital-composite good K from factor markets. The capital composite good is competitively assembled by an assembling firm using an energy-capital composite EK and general capital KR. The energy-capital composite is assembled from a renewable-energy-capital composite EKC and a fossil-energy-capital composite EKD. The fossil-energy-capital composite is again assembled from three different capital composites: oil ($EKD1$), gas ($EKD2$), and coal ($EKD3$). Each of the four energy-capital composites, EKC, $EKD1$, $EKD2$, and $EKD3$, is itself built by combining raw energy of the specific source Ei and specific capital Ki with $i \in \{C, D1, D2, D3\}$. Raw energy can be sourced at a fixed price p^i (either imported or produced using a domestic, linear technology) and is subject to a unit tax τ^i.

The production function is modeled accordingly in nested CES form, as summarized in Figure 6.1. A CES aggregator is defined as follows:

[61] More detailed models also assume that energy is not only used in the production of a final good but is additionally directly consumed by households.

$$y = CES(x_i; i \in \mathcal{Y}) = \left(\sum_{i \in \mathcal{Y}} \alpha_i^{1-\rho^y} (e_i \cdot x_i)^{\rho^y} \right)^{1/\rho^y}, \tag{227}$$

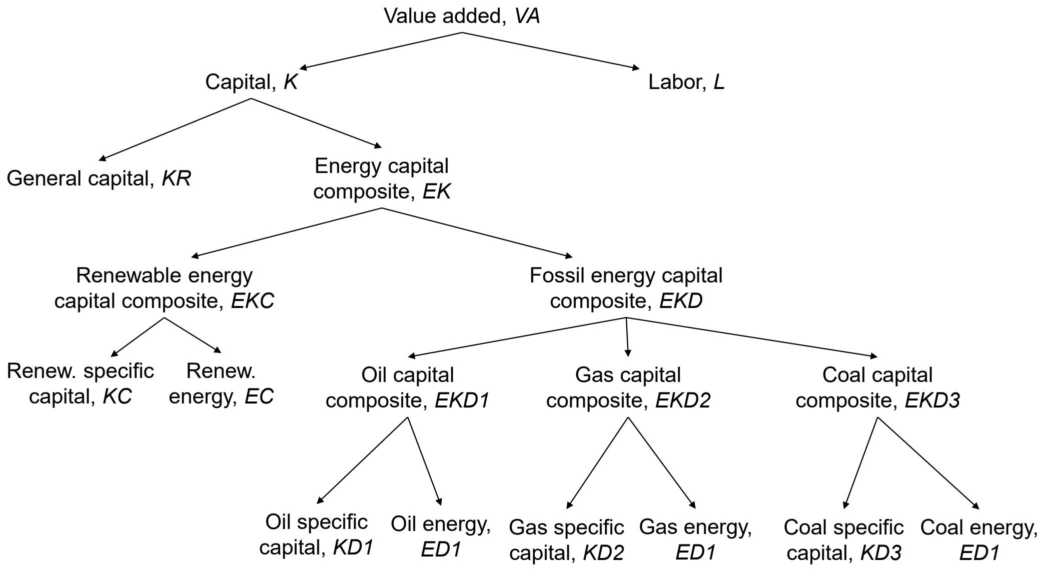
with $\mathcal{Y}$ being the set of inputs, α_i the corresponding share parameters, and $\sigma^y = 1/(1-\rho^Y)$ the elasticity of substitution. The parameters e_i are efficiency coefficients that can be time-dependent and used to capture trends in the relative difference in technological advances of different energy sources.[62] The corresponding first-order conditions of the competitive assembling firms are static and simply given as (dropping time indices):

$$\begin{gathered}
p^h \cdot \frac{\partial VA}{\partial K} = p^K \cdot (1+\tau^K), \quad p^h \cdot \frac{\partial VA}{\partial L} = w \cdot (1+\tau^F), \\
p^K \cdot \frac{\partial K}{\partial KR} = p^{KR}, \quad p^K \cdot \frac{\partial K}{\partial EK} = p^{EK}, \\
p^{EK} \cdot \frac{\partial EK}{\partial EKC} = p^{EKC}, \quad p^{EK} \cdot \frac{\partial EK}{\partial EKD} = p^{EKD}, \\
p^{EKC} \cdot \frac{\partial EKC}{\partial KC} = p^{KC}, \quad p^{EKC} \cdot \frac{\partial EK}{\partial EC} = p^{EC} + \tau^{EC}, \\
p^{EKD} \cdot \frac{\partial EKD}{\partial EKD1} = p^{EKD1}, \; p^{EKD} \cdot \frac{\partial EKD}{\partial EKD2} = p^{EKD2}, \; p^{EKD} \cdot \frac{\partial EKD}{\partial EKD3} = p^{EKD3}, \\
p^{EKD1} \cdot \frac{\partial EKD1}{\partial KD1} = p^{KD1}, \quad p^{EKD1} \cdot \frac{\partial EKD1}{\partial ED1} = p^{ED1} + \tau^{ED1}, \\
p^{EKD2} \cdot \frac{\partial EKD2}{\partial KD2} = p^{KD2}, \quad p^{EKD2} \cdot \frac{\partial EKD2}{\partial ED2} = p^{ED2} + \tau^{ED2}, \\
p^{EKD3} \cdot \frac{\partial EKD3}{\partial KD3} = p^{KD3}, \quad p^{EKD3} \cdot \frac{\partial EKD3}{\partial ED3} = p^{ED3} + \tau^{ED3}.
\end{gathered} \tag{228}$$

There are five different specific capital stocks in this setting: KR, KC, $KD1$, $KD2$, and $KD3$. Each is accumulated by a capital goods firm, identical to the set-up in section 2.11.3. Changes in relative prices or relative efficiency result in a changed energy mix as well as an endogenous shift in the economy's capital-energy intensity (E/K with $E = \sum_i Ei$). The reason for modeling energy source-specific capital stocks is to make these adjustments sluggish. The resulting consumption of energy in physical units by source can then be used to compute CO_2 emissions, simply by multiplying with the corresponding emission coefficients. The result of the energy sub-model is linked to the core model via three connections: changes in the capital-energy intensity (E/K), changes in the relative energy consumption shares (Ei/E), and changes to the price of capital p^K, which are fed into the core model, which employs a production function just taking labor and capital as inputs via a price of capital wedge: $\tilde{\tau}^K$. The wedge enters the optimality condition of the final goods

[62]The efficiency coefficients can be integrated into the share parameters. For the top-level aggregator $VA(K, L)$ the efficiency coefficients are always 1.

Figure 6.1: Incorporating Energy in the Production Function



Source: own illustration.

producers just as a tax on capital. Once energy consumption is determined, tax revenue on energy and emissions[63] can be simply computed. The taxes paid are then deducted lump-sum in the resource constraints of the final goods producers.[64]

6.1.2 Other Channels

Other channels added to the module comprise damages, emission-related penalty payments, and other climate measures. The amount of climate-change-related damages is derived from external sources. In the model, they are converted into higher depreciation rates of the private and public capital stock. Emission-related penalty payments (e.g., subject to the Effort Sharing Regulation at the EU level) are based on the derived emissions and are modeled as a capital transfer from the government to abroad. Lastly, there is the option to incorporate miscellaneous climate protection measures by manually shocking public costs and changes in the energy efficiency coefficients on a case-by-case basis.

[63]Taxes on emissions can be converted into taxes on the consumption of energy by applying the corresponding emission coefficients.

[64]The code for solving the energy sub-model is available online (`energy_production.Rmod` in `https://github.com/RmodPkg/RmodExamples`). It was implemented using a toolbox for generically solving dynamic, discrete time-models in R/C++: `https://github.com/RmodPkg/Rmod`.

Appendix

A Proofs

Proof of lemma 2.1 - Consumption function (no income effect specification)

Proof. The idea of the proof is to consecutively insert the Euler equation into the intertemporal budget constraint. Because of habit persistence, the proof is slightly more involved. We first have to rewrite the intertemporal budget constraint in terms of $\tilde{C}$ instead of C. Note that, in order to save notation, we drop the U-superscript for unconstrained households, as well as the skill superscript. Recall that $\tilde{C}_t^a = C_t^a - \kappa \bar{C}_{t-1}^{a-1}$, where $\bar{C}$ is average consumption. This implies that the household does not recognize that changing consumption today will also change the consumption anchor tomorrow. This simplifies the Euler equation considerably. Ex-post, we have that $\bar{C}_t^a = C_t^a$, which is automatically inserted after optimization. Proceed as follows.

Rewriting the budget constraint

First, express the intertemporal budget constraint (11) to be forward-looking.

$$A_t^a = p_t^C C_t^a - \bar{y}_t^a + \mathcal{G}_t A_{t+1}^{a+1} / \bar{R}_t^W, \tag{229}$$

Now, insert recursively to get

$$A_t^a = p_t^C C_t^a - \bar{y}_t^a + \sum_{s=t+1}^{t+\bar{a}-a} \left[p_s^C C_s^{a+s-t} - \bar{y}_s^{a+s-t} \right] \prod_{u=t}^{s-1} \frac{\mathcal{G}_u}{\bar{R}_u^W}. \tag{230}$$

Expressed in long form, this is

$$
\begin{aligned}
A_t^a = {} & p_t^C C_t^a - \bar{y}_t^a - \sum_{s=t+1}^{t+\bar{a}-a} \bar{y}_s^{a+s-t} \prod_{u=t}^{s-1} \frac{\mathcal{G}_u}{\bar{R}_u^W} \\
& + p_{t+1}^C C_{t+1}^{a+1} \frac{\mathcal{G}_t}{\bar{R}_t^W} \\
& + p_{t+2}^C C_{t+2}^{a+2} \frac{\mathcal{G}_t \mathcal{G}_{t+1}}{\bar{R}_t^W \bar{R}_{t+1}^W} \\
& + \dots \\
& + p_{t+\bar{a}-a-1}^C C_{t+\bar{a}-a-1}^{\bar{a}-1} \frac{\mathcal{G}_t \mathcal{G}_{t+1} \dots \mathcal{G}_{t+\bar{a}-a-2}}{\bar{R}_t^W \bar{R}_{t+1}^W \dots \bar{R}_{t+\bar{a}-a-2}^W} \\
& + p_{t+\bar{a}-a}^C C_{t+\bar{a}-a}^{\bar{a}} \frac{\mathcal{G}_t \mathcal{G}_{t+1} \dots \mathcal{G}_{t+\bar{a}-a-2} \mathcal{G}_{t+\bar{a}-a-1}}{\bar{R}_t^W \bar{R}_{t+1}^W \dots \bar{R}_{t+\bar{a}-a-2}^W \bar{R}_{t+\bar{a}-a-1}^W}
\end{aligned}
$$

Just focus on the last consumption entry and replace it using (31), i.e., $C_{t+\bar{a}-a}^{\bar{a}} = \tilde{C}_{t+\bar{a}-a}^{\bar{a}} + \kappa C_{t+\bar{a}-a-1}^{\bar{a}-1}$. For the last two lines, this implies

$$
\begin{aligned}
A_t^a = {} & \dots \\
& + p_{t+\bar{a}-a-1}^C C_{t+\bar{a}-a-1}^{\bar{a}-1} \left[1 + \kappa \frac{p_{t+\bar{a}-a}^C}{p_{t+\bar{a}-a-1}^C} \frac{\mathcal{G}_t}{\bar{R}_{t+\bar{a}-a}^W} \right] \frac{\mathcal{G}_t \mathcal{G}_{t+1} \dots \mathcal{G}_{t+\bar{a}-a-2}}{\bar{R}_t^W \bar{R}_{t+1}^W \dots \bar{R}_{t+\bar{a}-a-2}^W} \\
& + p_{t+\bar{a}-a}^C \tilde{C}_{t+\bar{a}-a}^{\bar{a}} \frac{\mathcal{G}_t \mathcal{G}_{t+1} \dots \mathcal{G}_{t+\bar{a}-a-2} \mathcal{G}_{t+\bar{a}-a-1}}{\bar{R}_t^W \bar{R}_{t+1}^W \dots \bar{R}_{t+\bar{a}-a-2}^W \bar{R}_{t+\bar{a}-a-1}^W}.
\end{aligned}
$$

Let us define the following

$$
\Gamma_t^a = 1 + \kappa \left[\frac{p_{t+1}^C}{p_t^C} \frac{\mathcal{G}_t}{\bar{R}_t^W} \right] \Gamma_{t+1}^{a+1}, \quad \text{with} \quad \Gamma_t^{\bar{a}} = 1, \tag{231}
$$

which would trivially be 1 for all times and age groups if $\kappa = 0$. Observe that the budget constraint can now be written as

$$
\begin{aligned}
A_t^a = {} & p_t^C \Gamma_t^a \tilde{C}_t^a + p_t^C \Gamma_t^a \kappa C_{t-1}^{a-1} - \bar{y}_t^a - \sum_{s=t+1}^{t+\bar{a}-a} \bar{y}_s^{a+s-t} \prod_{u=t}^{s-1} \frac{\mathcal{G}_u}{\bar{R}_u^W} \\
& + p_{t+1}^C \Gamma_{t+1}^{a+1} \tilde{C}_{t+1}^{a+1} \frac{\mathcal{G}_t}{\bar{R}_t^W} \\
& + p_{t+2}^C \Gamma_{t+2}^{a+2} \tilde{C}_{t+2}^{a+2} \frac{\mathcal{G}_t \mathcal{G}_{t+1}}{\bar{R}_t^W \bar{R}_{t+1}^W} \\
& + \dots \\
& + p_{t+\bar{a}-a-1}^C \Gamma_{t+\bar{a}-a-1}^{\bar{a}-1} \tilde{C}_{t+\bar{a}-a-1}^{\bar{a}-1} \frac{\mathcal{G}_t \mathcal{G}_{t+1} \dots \mathcal{G}_{t+\bar{a}-a-2}}{\bar{R}_t^W \bar{R}_{t+1}^W \dots \bar{R}_{t+\bar{a}-a-2}^W} \\
& + p_{t+\bar{a}-a}^C \Gamma_{t+\bar{a}-a}^{\bar{a}} \tilde{C}_{t+\bar{a}-a}^{\bar{a}} \frac{\mathcal{G}_t \mathcal{G}_{t+1} \dots \mathcal{G}_{t+\bar{a}-a-2} \mathcal{G}_{t+\bar{a}-a-1}}{\bar{R}_t^W \bar{R}_{t+1}^W \dots \bar{R}_{t+\bar{a}-a-2}^W \bar{R}_{t+\bar{a}-a-1}^W}
\end{aligned}
$$

This conceptually resembles the original budget constraint, except for two things. First, every consumption term is preceded by a Γ term, and there is a remainder of $p_t^C \Gamma_t^a \kappa C_{t-1}^{a-1}$. For $a = \underline{a}$, this term vanishes. In short, the budget constraint can now be written as

$$
A_t^a = p_t^C \Gamma_t^a \tilde{C}_t^a + p_t^C \Gamma_t^a \kappa C_{t-1}^{a-1} - \bar{y}_t^a + \sum_{s=t+1}^{t+\bar{a}-a} \left[p_s^C \Gamma_s^{a+s-t} \tilde{C}_s^{a+s-t} - \bar{y}_s^{a+s-t} \right] \prod_{u=t}^{s-1} \frac{\mathcal{G}_u}{\bar{R}_u^W}.
$$

Extend this expression by Ψ_t^a and rewrite the budget constraint in terms of $Q_t^a \equiv \tilde{C}_t^a - \Psi_t^a$.

$$
\begin{aligned}
A_t^a = {} & p_t^C \Gamma_t^a Q_t^a + p_t^C \Gamma_t^a \Psi_t^a + p_t^C \Gamma_t^a \kappa C_{t-1}^{a-1} - \bar{y}_t^a \\
& + \sum_{s=t+1}^{t+\bar{a}-a} \left[p_s^C \Gamma_s^{a+s-t} Q_s^{a+s-t} + p_s^C \Gamma_s^{a+s-t} \Psi_s^{a+s-t} - \bar{y}_s^{a+s-t} \right] \prod_{u=t}^{s-1} \frac{\mathcal{G}_u}{\bar{R}_u^W}.
\end{aligned}
$$

Define human wealth as

$$
H_t^a \equiv \bar{y}_t^a - p_t^C \Gamma_t^a \Psi_t^a + \sum_{s=t+1}^{t+\bar{a}-a} \left[\bar{y}_s^{a+s-t} - p_s^C \Gamma_s^{a+s-t} \Psi_s^{a+s-t} \right] \prod_{u=t}^{s-1} \frac{\mathcal{G}_u}{\bar{R}_u^W}, \tag{232}
$$

Define total wealth $\mathcal{W}_t^a$ as the remainder such that $\mathcal{W}_t^a = A_t^a + H_t^a$, i.e.

$$
\mathcal{W}_t^a = p_t^C \Gamma_t^a Q_t^a + p_t^C \Gamma_t^a \kappa C_{t-1}^{a-1} + \sum_{s=t+1}^{t+\bar{a}-a} \left[p_s^C \Gamma_s^{a+s-t} Q_s^{a+s-t} \right] \prod_{u=t}^{s-1} \frac{\mathcal{G}_u}{\bar{R}_u^W}.
$$

Inserting the Euler equation

Insert the Euler equation (57) consecutively to express total wealth in terms of Q_t^a.

$$\begin{aligned}
\mathcal{W}_t^a &= p_t^C \Gamma_t^a Q_t^a + p_t^C \Gamma_t^a \kappa C_{t-1}^{a-1} + p_t^C Q_t^a \times \\
&\quad \left(\sum_{s=t+1}^{t+\bar{a}-a} \Gamma_s^{a+s-t} \prod_{u=t}^{s-1} \left(\bar{R}_u^W \frac{p_u^C}{p_{u+1}^C} \right)^{\sigma-1} \left(\hat{\beta}_u^{a+u-t} \gamma_u^{a+u-t} \right)^{\sigma} \right) \\
&= p_t^C Q_t^a \Omega_t^a + p_t^C \Gamma_t^a \kappa C_{t-1}^{a-1} \quad \text{where} \\
\Omega_t^a &= \Gamma_t^a + \left(\sum_{s=t+1}^{t+\bar{a}-a} \Gamma_s^{a+s-t} \prod_{u=t}^{s-1} \left(\bar{R}_u^W \frac{p_u^C}{p_{u+1}^C} \right)^{\sigma-1} \left(\hat{\beta}_u^{a+u-t} \gamma_u^{a+u-t} \right)^{\sigma} \right).
\end{aligned}$$

Hence, the consumption function in terms of Q_t^a is

$$Q_t^a = \left(\Omega_t^a p_t^C \right)^{-1} \left[\mathcal{W}_t^a - p_t^C \Gamma_t^a \kappa C_{t-1}^{a-1} \right] = \left(\Omega_t^a p_t^C \right)^{-1} \left(A_t^a + H_t^a - p_t^C \Gamma_t^a \kappa C_{t-1}^{a-1} \right).$$

and consequently, the consumption function in terms of C_t^a is

$$C_t^a = \left(\Omega_t^a p_t^C \right)^{-1} \left(A_t^a + H_t^a - p_t^C \Gamma_t^a \kappa C_{t-1}^{a-1} \right) + \kappa C_{t-1}^{a-1} + \Psi_t^a. \tag{233}$$

The consumption function for the first age group $a = \underline{a}$ collapses to

$$C_t^{\underline{a}} = \left(\Omega_t^{\underline{a}} p_t^C \right)^{-1} H_t^{\underline{a}} + \Psi_t^{\underline{a}}, \tag{234}$$

as assets at the beginning of life and consumption in the period before are zero. In a steady state, the consumption functions can be solved forward in age by starting from $a = \underline{a}$. The forward-looking recursive solution block is given by

$$\Gamma_t^a = 1 + \kappa \left[\frac{p_{t+1}^C}{p_t^C} \frac{\mathcal{G}_t}{\bar{R}_t^W} \right] \Gamma_{t+1}^{a+1}, \quad \text{with} \quad \Gamma_t^{\bar{a}} = 1, \tag{235}$$

$$\Omega_t^a = \Gamma_t^a + \left(\gamma_t^a \hat{\beta}_t^a \right)^{\sigma} \left(\bar{R}_t^W \frac{p_t^C}{p_{t+1}^C} \right)^{\sigma-1} \Omega_{t+1}^{a+1}, \tag{236}$$

$$H_t^a = \bar{y}_t^a - p_t^C \Gamma_t^a \Psi_t^a + \frac{\mathcal{G}_t H_{t+1}^{a+1}}{\bar{R}_t^W}. \tag{237}$$

■

Proof of lemma 2.2 - Aggregate assets

Proof. The proof is presented for a model with only one skill class. Generalizing to more skill classes is straightforward and therefore omitted. Recall that constrained households do not play a role here, as they do not have assets. Multiply (11) by $N_t^{U,a}\gamma_t^a$, use the demographic law of motion (2), sum over a, eliminate $\sum_{a=\underline{a}}^{\overline{a}} ab_t^a N_t^{U,a}$ using (67), and use $\sum_{a=\underline{a}}^{\overline{a}} iv_t^a N_t^{U,a} = 0$:

$$\begin{aligned}
\mathcal{G}_t A_{t+1}^{a+1} N_t^{U,a} &= \bar{R}_t^W S_t^a N_t^{U,a} \Leftrightarrow \\
\mathcal{G}_t \mathcal{N}_t A_{t+1}^{a+1} N_{t+1}^{U,a+1} &= \bar{R}_t^W \gamma_t^a S_t^a N_t^{U,a} + \bar{R}_t^W S_t^a Mig_{t+1}^{U,a+1} \mathcal{N}_t \Leftrightarrow \\
\hat{\mathcal{G}}_t A_{t+1}^{a+1} N_{t+1}^{U,a+1} &= \bar{R}_t^W \left(S_t^a - (1-\gamma_t^a) S_t^a\right) N_t^{U,a} + \bar{R}_t^W A_t^{Mig} \Rightarrow \\
\hat{\mathcal{G}}_t \sum_{a=\underline{a}}^{\overline{a}} A_{t+1}^{a+1} N_{t+1}^{U,a+1} &= \bar{R}_t^W \sum_{a=\underline{a}}^{\overline{a}} \left(S_t^a - (1-\gamma_t^a) S_t^a\right) N_t^{U,a} + \bar{R}_t^W A_t^{Mig} \Leftrightarrow \\
\hat{\mathcal{G}}_t \sum_{a=\underline{a}}^{\overline{a}} A_{t+1}^{a+1} N_{t+1}^{U,a+1} &= \bar{R}_t^W \sum_{a=\underline{a}}^{\overline{a}} S_t^a N_t^{U,a} - \bar{R}_t^W \sum_{a=\underline{a}}^{\overline{a}} ab_t^a N_t^{U,a} + \bar{R}_{t+1}^W A_t^{Mig} \Leftrightarrow \\
\hat{\mathcal{G}}_t \sum_{a=\underline{a}}^{\overline{a}} A_{t+1}^{a+1} N_{t+1}^{U,a+1} &= \bar{R}_t^W \sum_{a=\underline{a}}^{\overline{a}} \left[A_t^a + y_t^a + iv_t^a - Z_t^{U,a} - p_t^C C_t^{U,a}\right] N_t^{U,a} \\
&\quad + \bar{R}_t^W A_t^{Mig} \Leftrightarrow \\
\hat{\mathcal{G}}_t \sum_{a=\underline{a}}^{\overline{a}} A_{t+1}^{a+1} N_{t+1}^{U,a+1} &= \bar{R}_t^W \left[A_t + A_t^{Mig} - Z_t^U + y_t^U - p_t^C C_t^U\right]
\end{aligned}$$

The left-hand side can be rearranged as follows, using the fact that new entrants into adulthood[65] have zero assets, and that the mass of $N_t^{U,\overline{a}+1}$ is always zero, as people with age $\overline{a}$ die with certainty.

$$\begin{aligned}
\hat{\mathcal{G}}_t \sum_{a=\underline{a}}^{\overline{a}} A_{t+1}^{a+1} N_{t+1}^{U,a+1} &= \hat{\mathcal{G}}_t \sum_{a=\underline{a}-1}^{\overline{a}-1} A_{t+1}^{a+1} N_{t+1}^{U,a+1} + \hat{\mathcal{G}}_t \underbrace{A_{t+1}^{\overline{a}+1} N_t^{U,\overline{a}+1}}_{=0} - \hat{\mathcal{G}}_t \underbrace{A_{t+1}^{\underline{a}-1} N_t^{U,\underline{a}-1}}_{=0} \\
&= \hat{\mathcal{G}}_t \sum_{a=\underline{a}-1}^{\overline{a}-1} A_{t+1}^{a+1} N_{t+1}^{U,a+1} = \hat{\mathcal{G}}_t \sum_{a=\underline{a}}^{\overline{a}} A_{t+1}^{a} N_{t+1}^{U,a} = \hat{\mathcal{G}}_t A_{t+1}
\end{aligned}$$

Hence,

$$\hat{\mathcal{G}}_t A_{t+1} = \bar{R}_t^W \left[A_t + A_t^{Mig} - Z_t^U + y_t^U - p_t^C C_t^U\right], \tag{238}$$

[65] Recall that all children can be neglected, as they do not hold assets.

or after adding the consumption of the constrained households ($C_t^C = (y_t^C - Z_t^C)/p_t^C$),

$$\hat{\mathcal{G}}_t A_{t+1} = \bar{R}_t^W \left[A_t + A_t^{Mig} - Z_t + y_t - p_t^C C_t \right].$$

∎

Proof of theorem 2.1 - Hayashi's Theorem

Proof. Take the envelope condition for K_t (125), multiply both sides by K_t, and expand the right-hand side by $\frac{q_{t+1}}{R_t^{V,h}} I_t$.

$$\begin{aligned} q_t K_t &= (1-\tau_t^{prof}) p_t^K K_t - (1-\phi_0^\tau \tau^{prof}) p_t^I J_{K_t} K_t + \phi_0^\tau \tau_t^{prof} p_t^I \delta^K K_t \\ &\quad + \frac{q_{t+1}}{R_t^{V,h}} \left[(1-\delta^K) K_t + I_t \right] - \frac{q_{t+1}}{R_t^{V,h}} I_t, \\ q_t K_t &= (1-\tau_t^{prof}) p_t^K K_t - (1-\phi_0^\tau \tau_t^{prof}) p_t^I J_{K_t} K_t + \phi_0^\tau \tau_t^{prof} p_t^I \delta^K K_t \\ &\quad + \frac{q_{t+1}}{R_t^{V,h}} \hat{\mathcal{G}}_t K_{t+1} - \left[1 - sub_t^I + (1-\phi_0^\tau \tau_t^{prof}) J_{I_t} \right] p_t^I I_t, \\ q_t K_t &= \chi_t - sub_t^l + \frac{q_{t+1}}{R_t^{V,h}} \hat{\mathcal{G}}_t K_{t+1}. \end{aligned}$$

From the first to the second equation, we used the law of motion (122) and the optimality condition for investment. From the second to the last equation, we used Euler's theorem, the linear homogeneity of the adjustment cost function, and the definition of per-period profits χ. Solving forward yields Hayashi (1982)'s result.

$$q_t K_t = \left[\chi_t + \sum_{s=t+1}^{\infty} \chi_s \prod_{u=t}^{s-1} \frac{\hat{\mathcal{G}}_u}{R_u} \right] - \left[sub_t^l + \sum_{s=t+1}^{\infty} sub_s^l \prod_{u=t}^{s-1} \frac{\hat{\mathcal{G}}_u}{R_u} \right] = V_t^C - V_t^R. \tag{239}$$

∎

Proof of theorem 2.2 - Walras' Law

Proof. First, we establish that

$$\hat{\mathcal{G}}_t \frac{A_{t+1}}{\bar{R}_t^W} = \hat{\mathcal{G}}_t \frac{A_{t+1}}{R_t^W} + T_t^R, \tag{240}$$

which comes from combining (71) and (72). For the proof of Walras' Law, proceed as follows. We start from the aggregate intertemporal budget constraint (71) extended by (155) and (156). First, insert (240). Then use (152) to eliminate A_t and replace $V_t^h \equiv V_t^C + V_t^F$.

$$\hat{\mathcal{G}}_t \frac{A_{t+1}}{\bar{R}_t^W} = A_t - \zeta_t^{IV} - \zeta_t^{AB} + A_t^{Mig} - Z_t + y_t - p_t^C C_t,$$
$$\hat{\mathcal{G}}_t \frac{A_{t+1}}{R_t^W} = A_t - \zeta_t^{IV} - \zeta_t^{AB} + A_t^{Mig} - Z_t + y_t - p_t^C C_t - T_t^R,$$
$$\hat{\mathcal{G}}_t \frac{A_{t+1}}{R_t^W} = D_t^F + D_t^G + V_t^h - \zeta_t^A - \zeta_t^{IV} - \zeta_t^{AB} + A_t^{Mig} - Z_t + y_t - p_t^C C_t - T_t^R.$$

Rewrite this by inserting V_t^C and V_t^F using (121) and (147) and eliminate χ_t and Π_t by using (120) and (104). Next, use the definition of T_t^{prof}. We use $j \in \mathcal{S} \times \{Y, O\}$ as the labor market index.

$$\begin{aligned}\hat{\mathcal{G}}_t \left[\frac{A_{t+1}}{R_t^W} - \frac{V_{t+1}^h}{R_t^{V,h}}\right] &= D_t^F + D_t^G - \zeta_t^A + p_t^K K_t - (1 - sub_t^I) p_t^I I_t - p_t^I J_t + sub_t^l \\ &+ (1 - \tau_t^Y) p_t^h Y_{i,t}^h - (1 + \tau_t^K) p_t^K K_{i,t} - (1 + \tau_t^F) \sum_j w_t^j L_{i,t}^j \\ &- cV_{i,t} - p_t^h \bar{f}_{i,t} + p_t^h sub_t^f \bar{f}_{i,t} + A_t^{Mig} - Z_t + y_t - p_t^C C_t \\ &- T_t^R - T_t^{prof} - \zeta_t^{IV} - \zeta_t^{AB}.\end{aligned}$$

Use the fact that the mass of variety producers is 1, and insert $K^D = K_i$, $L^{D,j} = L_i^j$, $Y^h = Y_i^h$, $\bar{f} = \bar{f}_i$, $V = V_i$. Then use the definitions of $\hat{Y}_t$, T^K, and ζ^K to arrive at

$$\begin{aligned}\hat{\mathcal{G}}_t \left[\frac{A_{t+1}}{R_t^W} - \frac{V_{t+1}^h}{R_t^{V,h}}\right] &= D_t^F + D_t^G - \zeta_t^A - p_t^K \zeta_t^K - (1 - sub_t^I) p_t^I I_t - p_t^I J_t + sub_t^l \\ &+ p_t^h \hat{Y}_t^h - \tau_t^Y p_t^h Y_t^h - (1 + \tau_t^F) \sum_j w_t^j L_t^{D,j} + p_t^h sub^f \bar{f}_t \\ &+ A_t^{Mig} - Z_t + y_t - p_t^C C_t - T_t^R - T_t^{prof} - T_t^K - \zeta_t^{IV} - \zeta_t^{AB}.\end{aligned}$$

Replace the asset term on the left-hand side using (240), insert the definition of T_t^Y, and rearrange to arrive at

$$\begin{aligned}\hat{\mathcal{G}}_t \left[\frac{A_{t+1}}{R_t^W} - \frac{V_{t+1}^h}{R_t^{V,h}}\right] &= D_t^F + D_t^G - \zeta_t^A - p_t^K \zeta_t^K - \zeta_t^{IV} - \zeta_t^{AB} + A_t^{Mig} - Z_t + y_t \\ &+ p_t^h \hat{Y}_t^h - p_t^I I_t - p_t^I J_t - p_t^C C_t - T_t^{prof} - T_t^K + sub_t^l \\ &+ sub_t^I p_t^I I_t - T_t^Y - T_t^R + sub^f \bar{f}_t - (1 + \tau_t^F) \sum_j w_t^j L_t^{D,j}.\end{aligned}$$

Insert for aggregate household income y_t, split labor supply according to labor types, insert $\zeta^{L,j}$ and T_t^L, use $p_t^C = (1+\tau_t^C)\tilde{p}_t^C$, and rearrange terms to arrive at

$$\begin{aligned}\hat{\mathcal{G}}_t\left[\frac{A_{t+1}}{R_t^W} - \frac{V_{t+1}^h}{R_t^{V,h}}\right] = &-\zeta_t^A - p_t^K\zeta_t^K - \sum_j w_t^j\zeta^{L,j} - \zeta_t^{IV} - \zeta_t^{AB} + A_t^{Mig}\\ &- Z_t + D_t^F + D_t^G + p_t^h\hat{Y}_t^h - p_t^I I_t - p_t^I J_t\\ &- \tilde{p}_t^C C_t - T_t^{prof} - T_t^K + sub_t^I p_t^I I_t + sub_t^l - T_t^Y - T_t^R\\ &+ sub^f\bar{f}_t - T_t^L - \tau_t^C\tilde{p}_t^C C_t + B_t + P_t - T_t^P - T_t^T.\end{aligned}$$

Now, use the definitions of Rev_t and Exp_t from (131) and (130), and insert (154).

$$\begin{aligned}\hat{\mathcal{G}}_t\left[\frac{A_{t+1}}{R_t^W} - \frac{V_{t+1}^h}{R_t^{V,h}}\right] = &-\zeta_t^A - p_t^K\zeta_t^K - \sum_j w_t^j\zeta^{L,j} - \zeta_t^{IV} - \zeta_t^{AB} + A_t^{Mig}\\ &- Z_t + D_t^F + D_t^G + p_t^h\hat{Y}_t^h - p_t^I I_t - p_t^I J_t - \tilde{p}_t^C C_t\\ &- Rev_t + \tau_t^{C^G}\tilde{p}_t^{C^G} C_t^G + Exp_t - Z_t^{gov} - \tilde{p}_t^{C^G} C_t^G - p_t^{I^G} I_t^G.\end{aligned}$$

Insert the definition of ζ_t^G and immediately replace PB_t using (143). Cancel taxes on government consumption and observe that D_t^G cancels out. Expand by TB_t.

$$\begin{aligned}\hat{\mathcal{G}}_t\left[\frac{A_{t+1}}{R_t^W} - \frac{V_{t+1}^h}{R_t^{V,h}} - \frac{D_{t+1}^G}{R_t^{G,h}}\right] = &-\zeta_t^A - p_t^K\zeta_t^K - \sum_j w_t^j\zeta^{L,j} - \zeta_t^{IV} - \zeta_t^{AB} - \zeta_t^G\\ &+ A_t^{Mig} - Z_t - Z_t^{gov} + D_t^F + TB_t - TB_t + p_t^h\hat{Y}_t^h\\ &- p_t^I I_t - p_t^I J_t - \tilde{p}_t^C C_t - \tilde{p}_t^{C^G} C_t^G - p_t^{I^G} I_t^G.\end{aligned}$$

Insert the law of motion for foreign assets (144).

$$\begin{aligned}\hat{\mathcal{G}}_t\left[\frac{A_{t+1}}{R_t^W} - \frac{V_{t+1}^h}{R_t^{V,h}} - \frac{D_{t+1}^G}{R_t^{G,h}} - \frac{D_{t+1}^F}{R_t^D}\right] = &-\zeta_t^A - p_t^K\zeta_t^K - \sum_j w_t^j\zeta^{L,j} - \zeta_t^{IV} - \zeta_t^{AB}\\ &- \zeta_t^G + p_t^h\hat{Y}_t^h - p_t^I I_t - p_t^I J_t - \tilde{p}_t^C C_t\\ &- \tilde{p}_t^{C^G} C_t^G - p_t^{I^G} I_t^G - TB_t.\end{aligned}$$

Next, split the demands accordingly, $p_t^I(I_t + J_t) = p_t^{I,h} I_t^h + p_t^{I,m} I_t^m$, etc., and use the definition of TB_t to get

$$\hat{\mathcal{G}}_t \left[\frac{A_{t+1}}{R_t^W} - \frac{V_{t+1}^h}{R_t^{V,h}} - \frac{D_{t+1}^G}{R_t^{G,h}} - \frac{D_{t+1}^F}{R_t^D} \right] = -\zeta_t^A - p_t^K \zeta_t^K - \sum_j w_t^j \zeta^{L,j} - \zeta_t^{IV} - \zeta_t^{AB} - \zeta_t^G$$
$$+ p_t^h \left[\hat{Y}_t^h - I_t^h - C_t^h - C_t^{G,h} - I_t^{G,h} - E_t^h \right].$$

Insert the definition of ζ_t^Y to get

$$\hat{\mathcal{G}}_t \left[\frac{A_{t+1}}{R_t^W} - \frac{V_{t+1}^h}{R_t^{V,h}} - \frac{D_{t+1}^G}{R_t^{G,h}} - \frac{D_{t+1}^F}{R_t^D} \right] = -\zeta_t^A - p_t^K \zeta_t^K - \sum_j w_t^j \zeta^{L,j}$$
$$- \zeta_t^{IV} - \zeta_t^{AB} - \zeta_t^G - p_t^h \zeta_t^Y. \tag{241}$$

Now, we focus just on the left-hand side of (241) and rewrite it by inserting $\hat{A}_t = \hat{\mathcal{G}}_t A_{t+1}/R_t^W$ and $\hat{D}_t^F = \hat{\mathcal{G}}_t D_{t+1}^F / R_t^D$, etc., and then using $\hat{A}_t = \hat{A}_t^{V,h} + \hat{A}_t^{V,m} + \hat{A}_t^{G,h} + \hat{A}_t^{G,m}$, and the definitions of ζ_{t+1}^V, ζ_{t+1}^{DG}, and ζ_{t+1}^{DF}.

$$\begin{aligned}
\hat{\mathcal{G}}_t \left[\frac{A_{t+1}}{R_t^W} - \frac{V_{t+1}^h}{R_t^{V,h}} - \frac{D_{t+1}^G}{R_t^{G,h}} - \frac{D_{t+1}^F}{R_t^D} \right] &= \hat{A}_t - \hat{D}_t^F - \hat{D}_t^G - \hat{V}_t^h \\
&= \hat{A}_t^{V,h} + \hat{A}_t^{V,m} + \hat{A}_t^{G,h} + \hat{A}_t^{G,m} \\
&\quad - \zeta_t^{DF} - \hat{A}_t^{V,m} + \hat{A}_t^{*V,h} - \hat{A}_t^{G,m} + \hat{A}_t^{*G,h} \\
&\quad - \zeta_t^{DG} - \hat{A}_t^{G,h} - \hat{A}_t^{*G,h} \\
&\quad - \zeta_t^{V} - \hat{A}_t^{V,h} - \hat{A}_t^{*V,h} \\
&= - \left[\zeta_t^{DF} + \zeta_t^{DG} + \zeta_t^V \right].
\end{aligned}$$

Insert this on the left-hand side of (241) to get

$$p_t^h \zeta_t^Y + p_t^K \zeta_t^K + \sum_j w_t^j \zeta_t^{L,j} + \zeta_t^A + \zeta_t^G + \zeta_t^{IV} + \zeta_t^{AB}$$
$$- \left[\zeta_t^{DF} + \zeta_t^{DG} + \zeta_t^V \right] = 0. \tag{242}$$

We can eliminate ζ_t^A by taking the following steps: start from the definition, expand each term by the corresponding interest factor, and then replace it with end-of-period values.

$$
\begin{aligned}
\zeta_t^A &= D_t^F + D_t^G + V_t^h - A_t, \\
\zeta_t^A &= R_{t-1}^D \frac{D_t^F}{R_{t-1}^D} + R_{t-1}^{G,h} \frac{D_t^G}{R_{t-1}^{G,h}} + R_{t-1}^{V,h} \frac{V_t^h}{R_{t-1}^{V,h}} - R_{t-1}^W \frac{A_t}{R_{t-1}^W}, \\
\hat{\mathcal{G}}_{t-1}\zeta_t^A &= R_{t-1}^D \hat{D}_{t-1}^F + R_{t-1}^{G,h} \hat{D}_{t-1}^G + R_{t-1}^{V,h} \hat{V}_{t-1}^h - R_{t-1}^W \hat{A}_{t-1}.
\end{aligned}
$$

Again, use $\hat{A}_{t-1} = \hat{A}_{t-1}^{V,h} + \hat{A}_{t-1}^{V,m} + \hat{A}_{t-1}^{G,h} + \hat{A}_{t-1}^{G,m}$, the definitions of ζ_t^V, ζ_t^G, and ζ_t^{DF}. Next, use the definitions of R_t^D and R_t^W. Then, cancel out terms.

$$
\begin{aligned}
\hat{\mathcal{G}}_{t-1}\zeta_t^A =& R_{t-1}^D \left[\zeta_{t-1}^{DF} + \hat{A}_{t-1}^{V,m} - \hat{A}_{t-1}^{*V,h} + \hat{A}_{t-1}^{G,m} - \hat{A}_{t-1}^{*G,h}\right] + R_{t-1}^{G,h}\left[\zeta_{t-1}^{DG} + \hat{A}_{t-1}^{G,h} - \hat{A}_{t-1}^{*G,h}\right] \\
&+ R_{t-1}^{V,h}\left[\zeta_{t-1}^{V} + \hat{A}_{t-1}^{V,h} + \hat{A}_{t-1}^{*V,h}\right] - R_{t-1}^W\left[\hat{A}_{t-1}^{V,h} + \hat{A}_{t-1}^{V,m} + \hat{A}_{t-1}^{G,h} + \hat{A}_{t-1}^{G,m}\right], \\
\hat{\mathcal{G}}_{t-1}\zeta_t^A =& R_{t-1}^D \zeta_{t-1}^{DF} + R_{t-1}^{V,m}\hat{A}_{t-1}^{V,m} - R_{t-1}^{V,h}\hat{A}_{t-1}^{*V,h} + R_{t-1}^{G,m}\hat{A}_{t-1}^{G,m} - R_{t-1}^{G,h}\hat{A}_{t-1}^{*G,h} \\
&+ R_{t-1}^{G,h}\zeta_{t-1}^{DG} + R_{t-1}^{G,h}\hat{A}_{t-1}^{G,h} + R_{t-1}^{G,h}\hat{A}_{t-1}^{*G,h} \\
&+ R_{t-1}^{V,h}\zeta_{t-1}^{V} + R_{t-1}^{V,h}\hat{A}_{t-1}^{V,h} + R_{t-1}^{V,h}\hat{A}_{t-1}^{*V,h} \\
&- R_{t-1}^{V,h}\hat{A}_{t-1}^{V,h} - R_{t-1}^{V,m}\hat{A}_{t-1}^{V,m} - R_{t-1}^{G,h}\hat{A}_{t-1}^{G,h} - R_{t-1}^{G,m}\hat{A}_{t-1}^{G,m} \\
\hat{\mathcal{G}}_{t-1}\zeta_t^A =& R_{t-1}^D \zeta_{t-1}^{DF} + R_{t-1}^{G,h}\zeta_{t-1}^{DG} + R_{t-1}^{V,h}\zeta_{t-1}^{V}.
\end{aligned}
$$

Insert this result into (242) to obtain Walras' Law in its dynamic form.

$$
\begin{aligned}
0 =& p_t^h \zeta_t^Y + p_t^K \zeta_t^K + \sum_j w_t^j \zeta_t^{L,j} + \zeta_t^G + \zeta_t^{IV} + \zeta_t^{AB} \\
&+ \left[R_{t-1}^D \zeta_{t-1}^{DF} + R_{t-1}^{G,h}\zeta_{t-1}^{DG} + R_{t-1}^{V,h}\zeta_{t-1}^{V}\right] / \hat{\mathcal{G}}_{t-1} - \left[\zeta_t^{DF} + \zeta_t^{DG} + \zeta_t^V\right].
\end{aligned} \tag{243}
$$

Consequently, the steady-state version of Walras' Law is

$$
\begin{aligned}
0 =& p^h \zeta^Y + p^K \zeta^K + \sum_j w^j \zeta^{L,j} + \zeta^G + \zeta^{IV} + \zeta^{AB} \\
&+ \frac{r^{V,h} - \hat{g}}{\hat{\mathcal{G}}}\zeta^V + \frac{r^{G,h} - \hat{g}}{\hat{\mathcal{G}}}\zeta^{DG} + \frac{r^D - \hat{g}}{\hat{\mathcal{G}}}\zeta^{DF}.
\end{aligned} \tag{244}
$$

■

Proof of lemma 5.1 - S2 indicator

Proof. First, note that the compound interest factor $\alpha_{i,j}$ can be decomposed as follows: $\alpha_{i,j} = \alpha_{i,z} \cdot \alpha_{z+1,j}$ for any z such that $i \leq z < j$. Without loss of generality, assume that the initial period is set to $\underline{t} = 1$. We define the $S2$ indicator as the share of each period's GDP, i.e.,

$$D_2 = Q_1 \left[D_1 - PB_1 - S2 \cdot GDP_1 \right]. \tag{245}$$

Recursively inserting up to period T gives

$$\begin{aligned} D_T = {} & \alpha_{1,T} \cdot D_1 \\ & - \alpha_{1,T} \cdot [PB_1 + S2 \cdot GDP_1] \\ & - \alpha_{2,T} \cdot [PB_2 + S2 \cdot GDP_2] \\ & \dots \\ & - \alpha_{T-1,T} \cdot [PB_{T-1} + S2 \cdot GDP_{T-1}] \\ & - \alpha_{T,T} \cdot [PB_T + S2 \cdot GDP_T] \end{aligned}$$

Divide by $\alpha_{1,T}$ to get.

$$\begin{aligned} D_T / \alpha_{1,T} = {} & D_1 \\ & - [PB_1 + S2 \cdot GDP_1] \\ & - [PB_2 + S2 \cdot GDP_2] / \alpha_{1,1} \\ & \dots \\ & - [PB_{T-1} + S2 \cdot GDP_{T-1}] / \alpha_{1,T-2} \\ & - [PB_T + S2 \cdot GDP_T] / \alpha_{1,T-1} \end{aligned}$$

In more compact writing, this is

$$\frac{D_T}{\alpha_{1,T}} = D_1 - \sum_{i=1}^{T} \frac{PB_i}{\alpha_{1,i-1}} - S2 \cdot \sum_{i=1}^{T} \frac{GDP_i}{\alpha_{1,i-1}} \tag{246}$$

with $\alpha_{1,0} = 1$. We now take the limit $T \to \infty$, use the transversality condition ($\lim_{T\to\infty} D_T/\alpha_{1,T} = 0$), and solve for $S2$.

$$S2 = \frac{D_1 - \sum_{i=1}^{\infty} \frac{PB_i}{\alpha_{1,i-1}}}{\sum_{i=1}^{\infty} \frac{GDP_i}{\alpha_{1,i-1}}}. \tag{247}$$

In an application, we typically define a cut-off period, after which everything is assumed to be constant. Let this period be z such that $PB_t = \overline{PB}$, $GDP_t = \overline{GDP}$, $Q_t = \overline{Q}$, $\forall t \geq z$. The infinite sum expression for PB can then be split as follows:

$$\sum_{i=1}^{\infty} \frac{PB_i}{\alpha_{1,i-1}} = \sum_{i=1}^{z} \frac{PB_i}{\alpha_{1,i-1}} + \overline{PB} \cdot \sum_{i=z+1}^{\infty} \frac{1}{\alpha_{1,i-1}} \tag{248}$$

Focus just on the last infinite sum.

$$\begin{aligned} \sum_{i=z+1}^{\infty} \frac{1}{\alpha_{1,i-1}} &= \frac{1}{\alpha_{1,z}} + \frac{1}{\alpha_{1,z+1}} + \frac{1}{\alpha_{1,z+2}} + \dots \\ &= \frac{1}{\alpha_{1,z-1}} \times \left[\frac{1}{\alpha_{z,z}} + \frac{1}{\alpha_{z,z+1}} + \frac{1}{\alpha_{z,z+2}} + \dots \right] \end{aligned}$$

As Q is constant after z, we have $\alpha_{z,z} = \overline{Q}$, $\alpha_{z,z+1} = \overline{Q}^2$, $\alpha_{z,z+2} = \overline{Q}^3, \dots$. Therefore, the square bracket in the last expression can be written as $\sum_{i=1}^{\infty} (1/\overline{Q})^i$. We can then use the well-known geometric series result $\sum_{i=1}^{\infty} \beta^i = \frac{1}{1-\beta} - 1 = \frac{\beta}{1-\beta}$ for $\beta \in [0, 1)$. This implies that

$$\sum_{i=z+1}^{\infty} \frac{1}{\alpha_{1,i-1}} = \frac{1}{\alpha_{1,z-1}} \times \frac{1}{\overline{q}}, \tag{249}$$

with $\overline{q} = \overline{Q} - 1$. Putting everything together gives the formula for $S2$ in the finite time approximation.

$$S2 = \frac{D_1 - \left[\sum_{i=1}^{z} \frac{PB_i}{\alpha_{1,i-1}} + \frac{\overline{PB}}{\overline{q} \cdot \alpha_{1,z-1}} \right]}{\left[\sum_{i=1}^{z} \frac{GDP_i}{\alpha_{1,i-1}} + \frac{\overline{GDP}}{\overline{q} \cdot \alpha_{1,z-1}} \right]}. \tag{250}$$

■

B Change Log

This change log provides a short implementation history of the main model features:

- Version 2.3:
  - shifted time-index for interest and conditional survival rates
  - nominal instead of real interest taxation
  - added endogenous average interest rate through debt roll-over mechanism
  - revised marital status flows
  - more flexible income base for surviors' pensions
  - stock-flow adjustments
  - added climate module extension
- Version 2.2 (Schuster, 2021):
  - dynamic fit to historical data instead of steady state calibration
  - added tax functions for labor income
  - added search unemployment
  - added widows/widowers and orphans
  - added utility of retirement
- Version 2.1:
  - reduced skill classes from 5 to 3
  - extended pension systems from 1 to 3
  - added possibility of income effects of labor supply
  - added different foreign trade regimes (small open, small (semi-)open, closed)
  - added household size shifter
  - detrended model by population
- Version 2.0:
  - replaced generalized Fair-Taylor algorithm by hybrid tatonnement algorithm

- Version 1.7:
  - added imperfect substitution between types of labor (young vs. old or native vs. foreign)
  - added imperfect substitution of assets (portfolio choice)
  - added productive public capital
- Version 1.4:
  - added 5 skill classes
  - added monopolistic competition
  - added hand-to-mouth consumers
  - added habit formation in consumption
  - added most of the public sector
  - steady state calibration to base year 2014
- Version 0.x: base model as developed for a Masters' course in "General Equilibrium Policy Evaluation" (Schuster, 2015).

C Applications

The model description presented so far is quite general, with many special cases nested within it. In an actual application, one has to choose certain specifications, and typically not all features are used. In addition, many model features were added over time, while others were abandoned. Elasticities, and other parameters, have been changed over time, and the choice depends on the application. This section documents the used specifications and model parameterization for past applications.

C.1 Effects of Refugee Migration

"We use a rich numerical OLG model of Auerbach-Kotlikoff type to simulate the long-run effects of refugee migration starting 2015 for a country with an aging society and a generous welfare system, namely Austria. The respective refugee cohorts are on average younger, less educated and less productive than both natives and the average migrant. The net fiscal contribution results from two opposing effects: a positive demographic effect which is counteracted by worse labor market outcomes. We robustly find that public debt is higher throughout the simulation horizon 2015–2060 compared to the baseline. We further analyze the group-specific welfare consequences resulting from differentiated wage effects" (Holler and Schuster, 2020).

The analysis is based on the FISK OLG Model version 1.7.[66] Table C.1 summarizes the key parameter choices and targets. In contrast to the standard specification of model v1.7, we use two sub-populations (Austrian residents and refugees) that can differ in their socioeconomic characteristics. In the application, we explore the effects of different degrees of substitutability (based on the young/old labor type distinction applied to native and foreign labor). The effects are computed as deviations from a baseline (no extraordinary refugee influx), which was mainly based on the results of the Ageing Report 2015 (European Commission, 2015). Figure C.1 exemplarily shows the main results for GDP, consumption, employment, wages, primary balance, and the debt level.

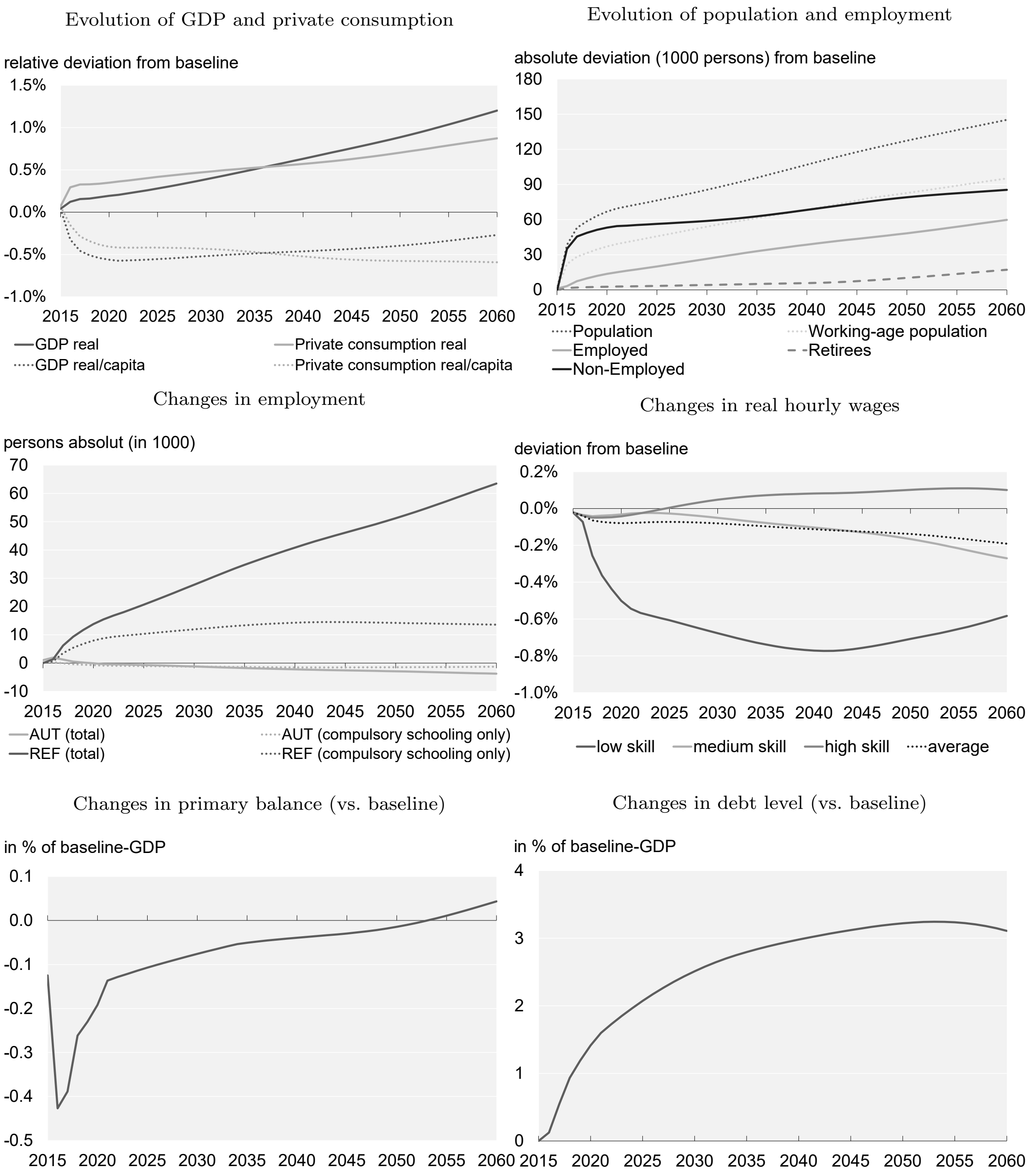
[66] An earlier version of the paper was published as a research report (Holler and Schuster, 2017) using the FISK OLG Model version 1.4.

Table C.1: Parameter overview used in Holler and Schuster (2020)

parameter	symbol	value	target/source
number of age groups	$\overline{a}+1$	99	modeling choice
productivity growth	g	0.014	Ageing Report 2015
discount rate	$(1-\beta)/\beta$	-0.035	peak in consumption between ages 60 and 70
depreciation rate	δ^K	0.03	match investment-to-GDP ratio
depreciation rate (publ. capital)	δ^G	0.03	same as private K
elasticity of intertemporal substitution	σ	0.25	Ratto et al. (2009)
elasticity of hours[1]	ε^ℓ	0.3	Chetty (2012)[2]
semi-elasticity of participation[1]	ε^δ	0.26	Chetty (2012)[2]
degree of habit formation	κ	0.3	Christoffel et al. (2008), Ratto et al. (2009)
Allen-Uzawa elasticities[3]	$\sigma^{L^{s=1,2,3},K}$	$\{1.4, 0.8, 0.5\}$	Krusell et al. (2000), Hassett (2002)
capital-adjust. cost	ϕ	5	halfway capital recovery after 8 years[4]
elasticity of subst. native/foreign labor	λ^{L^s}	10	Felbermayr et al. (2010)
elast. of subst. dom./foreign C	λ^C	1.2	Breuss et al. (2009), Ratto et al. (2009)
elast. of subst. dom./foreign C^G	λ^{C^G}	1.2	Breuss et al. (2009), Ratto et al. (2009)
elast. of subst. dom./foreign I	λ^I	1.2	Breuss et al. (2009), Ratto et al. (2009)
elast. of subst. dom./foreign I^G	λ^{I^G}	1.2	Breuss et al. (2009), Ratto et al. (2009)
elast. of subst. dom./foreign A^V	σ^{A^V}	1.2	same as for goods
elast. of subst. dom./foreign A^G	σ^{A^G}	1.2	same as for goods
elast. of subst. V/G assets	σ^A	5.0	Keuschnigg and Dietz (2007)
export price elasticity	λ^E	-2.5	Ratto et al. (2009)
foreign demand dom. assets A^V elast.	$\lambda^{V,h}$	∞	deactivated
foreign demand dom. assets A^G elast.	$\lambda^{G,h}$	∞	deactivated
import share C	ξ^C	0.30	input-output tables (STAT)
import share C^G	ξ^{C^G}	0.13	input-output tables (STAT)
import share I	ξ^I	0.40	input-output tables (STAT)
import share I^G	ξ^{I^G}	0.12	input-output tables (STAT)
love of variety (monop. comp.)	ϑ	10	Ratto et al. (2009)
efficiency public capital	σ^G	0.082	Bom and Ligthart (2014)
share of constrained households	$1-\pi^s$	$\{0.5, 0.4, 0.3, 0.2, 0.1\}$	Coenen et al. (2005), Ratto et al. (2009)
return rate on gov. bonds	r^G	0.014	closed interest-growth diff. ($r^G = g$)
return rate on firm bonds	r^V	0.04	$r^W \approx 0.03$ as in Ageing Report 2015
age-skill productivity profiles	θ^i	-	EU-SILC
age-skill taxes/transfers profiles	-	-	EU-SILC
age-skill C^G profiles	$c^{G,i}$	-	Hammer (2015)
fiscal shift parameters	-	-	match revenue/expend. (STAT)[5]
age-skill hours profiles	$\varphi_0^{\ell,i}$	-	EU-SILC
age-skill participation profiles	$\varphi_0^{\delta,i}$	-	RBLMS (STAT)

Notes: The calibration values chosen do not necessarily precisely match those of the sources cited, but are reasonably close. 1) No income effects specification (section 3.1). 2) Targeted Marshallian elasticities of around 0.2 based on Chetty (2012). 3) Nested CES-production function (Section 3.6.2). 4) See Cummins et al. (1996) and Radulescu and Stimmelmayr (2007). 5) Based on national accounts and national tax lists.

Figure C.1: Macroeconomic and fiscal effects due to refugee migration



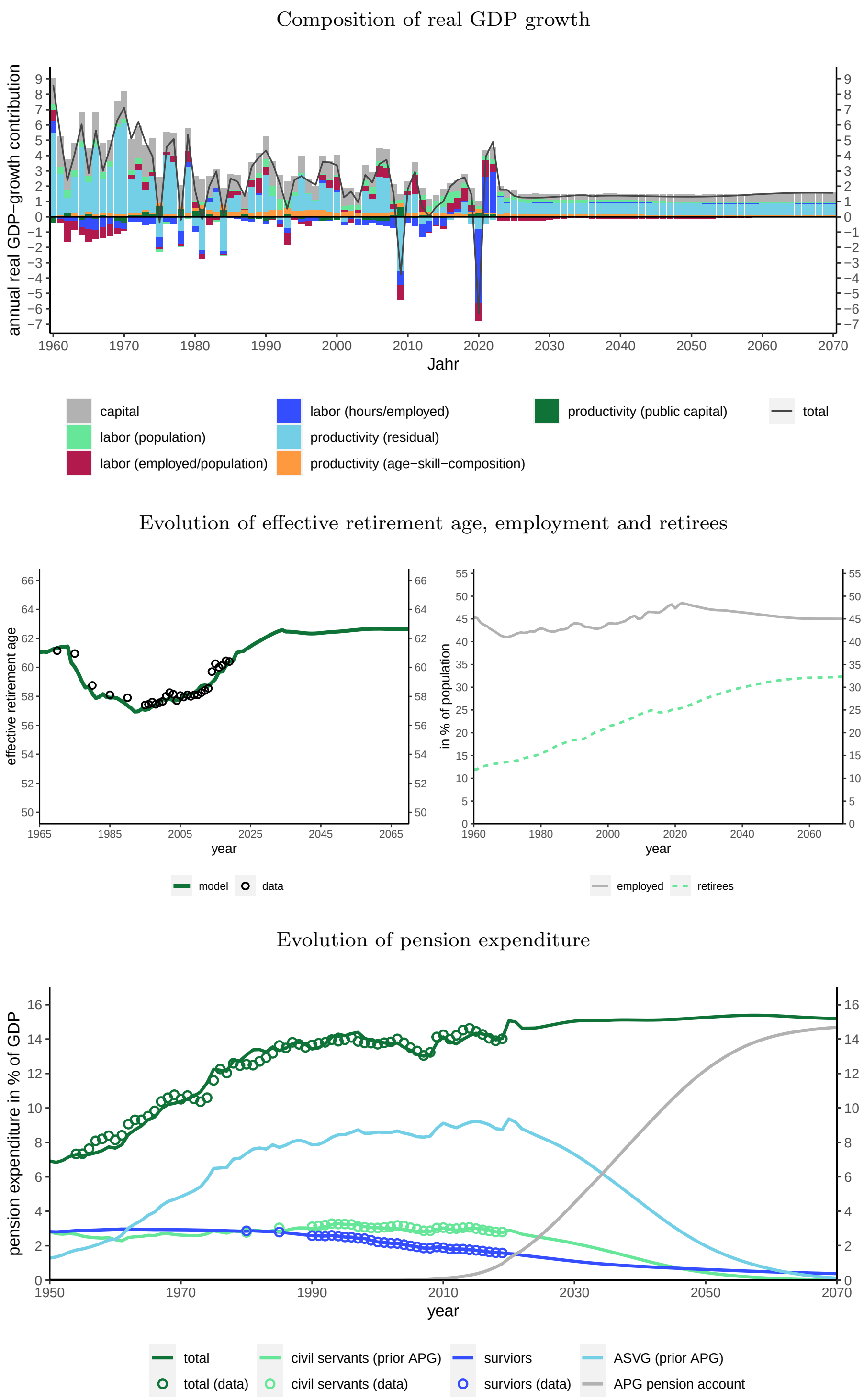
Source: Holler and Schuster (2020). Notes: Wages are measured as rates per efficiency unit. 'AUT'... domestic subpopulation, 'REF'... refugee subpopulation.

C.2 Fiscal Sustainability Report 2021

In 2021, the Austrian Fiscal Advisory Council published its first Fiscal Sustainability Report (FSR) (Fiscal Advisory Council, 2021), which from now on will be updated every three years. The main analysis was carried out with the FISK OLG Model version 2.2. The model is used to produce fiscal long-run projections and to quantify the fiscal space or fiscal gap over the projection horizon from 2021 to 2070, given a no-policy-change assumption. The analysis robustly finds that after the return of the primary balance to pre-pandemic levels, Austria will face a fiscal space that will slowly deteriorate and eventually turn into a fiscal gap. The deterioration is mainly due to increases in aging-related costs. The turning point at which the fiscal space will become a fiscal gap falls into the period from 2040 to the mid-2050s, depending on the assumptions. Next to various sensitivity scenarios, the report also presents simulation results on selected policy options, e.g., an increase in the effective retirement age in line with the increase in life expectancy, or a cap in unit cost trend growth of public health service expenditure. Figures C.2 to C.4 exemplarily show some of the results of the analysis.

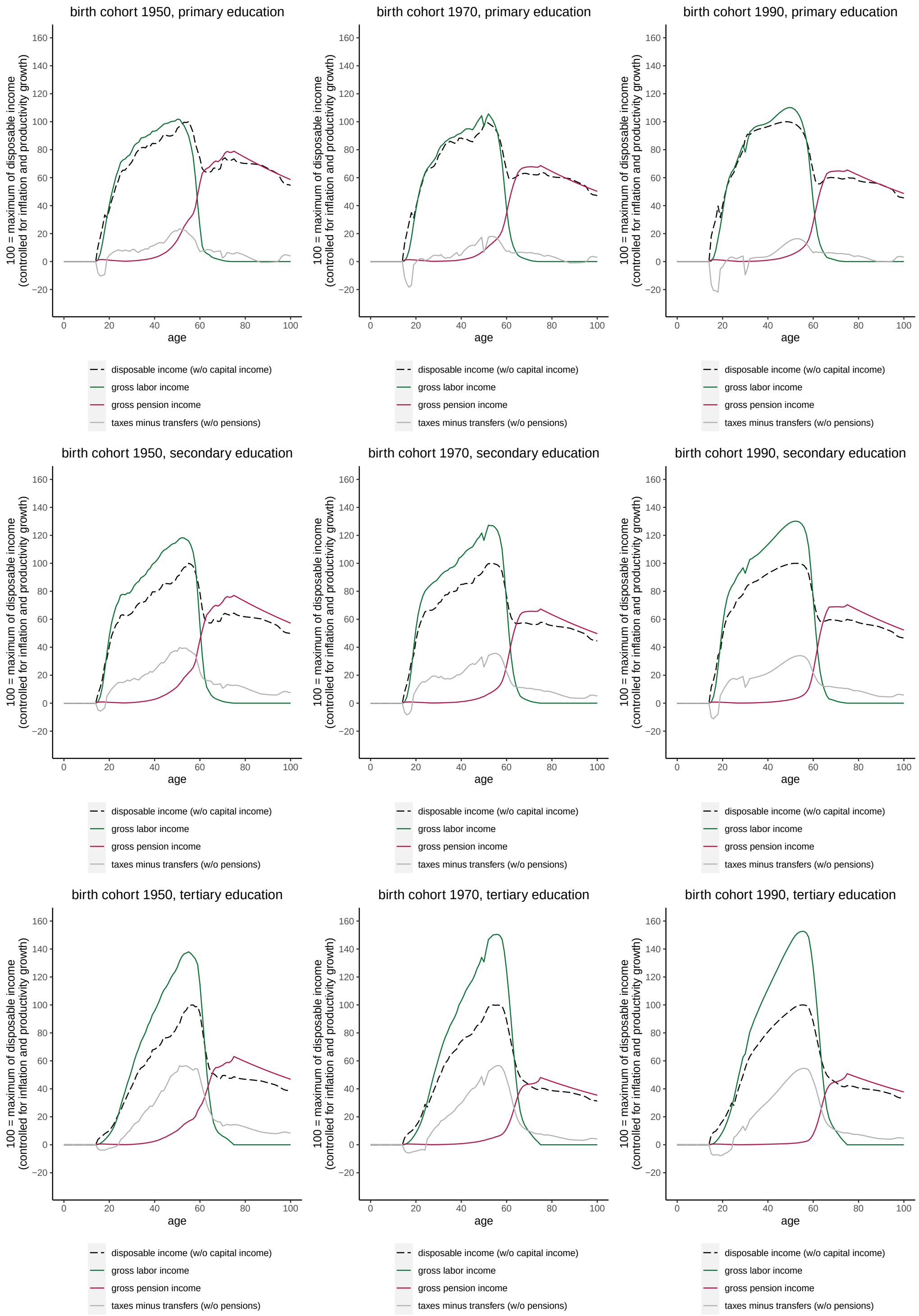
Table C.2 summarizes the parameter choices and targets. The FSR 2021 was the first application that relied on dynamically fitting the model to the historical data (see Section 5). To simplify this exercise, some of the model features have been deactivated (e.g., monopolistic competition, nested-CES production, vacancy posting). We use a semi-open economy specification by setting the 'risky' return rate $r^{V,h}$ in order to match the historical evolution of the current account and the net foreign asset position. Firm assets and government bonds are imperfect substitutes, and we assume that foreign demand for domestic government bonds is infinitely elastic, which allows us to set $r^{G,h}$ freely, i.e., to match the historical development of interest expenditure and to set an interest rate path for the future. Labor productivity growth is set to replicate the historical development of GDP per capita. The long-run average of 1.2% p.a. was used as the future labor productivity growth rate. Future trend growth in unit costs ('drift') of public consumption outlays (e.g., health care, education, long-term care (LTC)), and many transfers (e.g., health care, LTC, family) is derived from the historical fit. The COVID-19 pandemic is treated as an unanticipated shock to hours supply, productivity, unemployment, fiscal stimulus measures, and the virtual consumption tax $\tau^{C,pandemic}$ (which is immediately reimbursed as a lump-sum transfer following Eichenbaum et al., 2021, to match the consumption response).

Figure C.2: Main results of FSR 2021 for Austria 1/3



Source: Fiscal Advisory Council (2021)

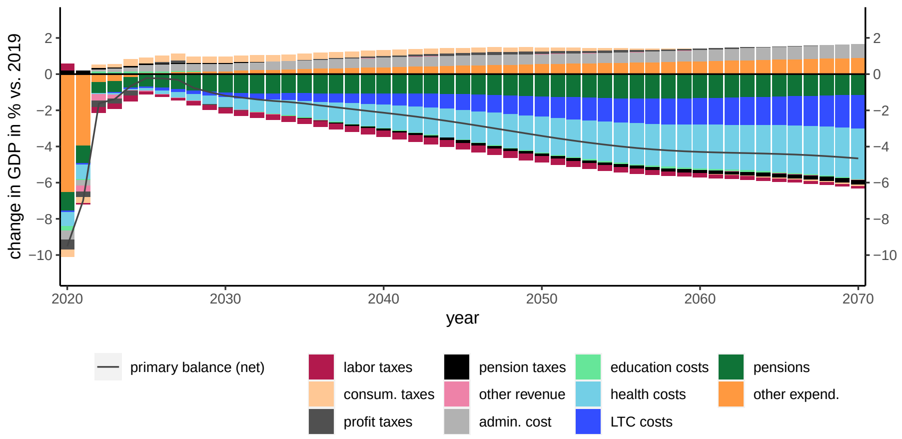
Figure C.3: Main results of FSR 2021 for Austria 2/3



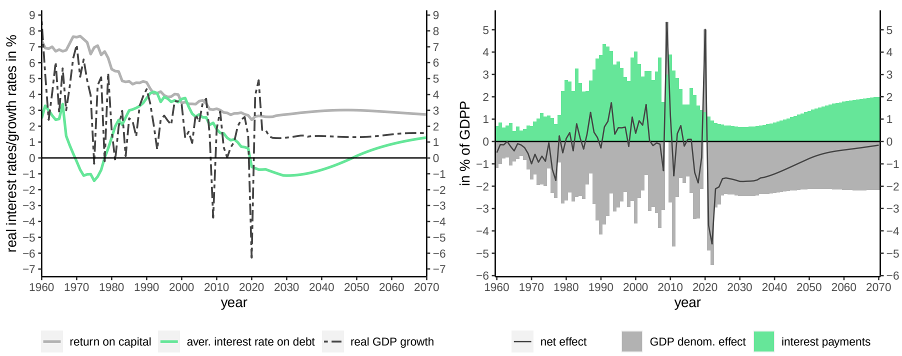
Source: Fiscal Advisory Council (2021)

Figure C.4: Main results of FSR 2021 for Austria 3/3

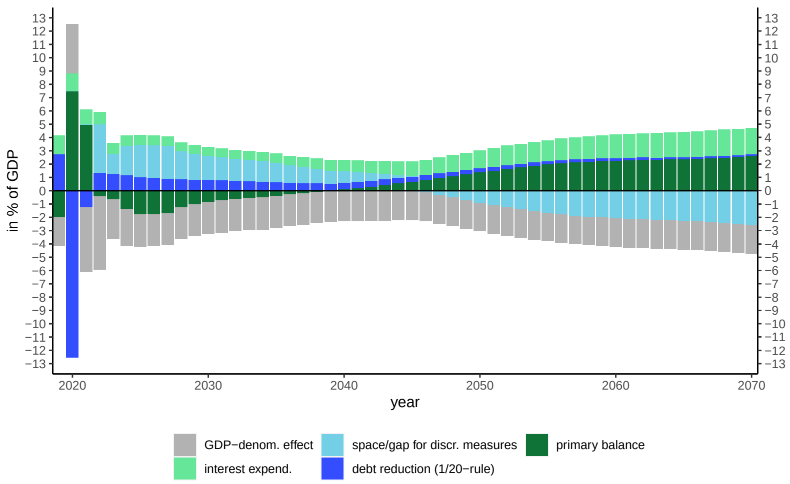
Change in primary balance versus 2019



Evolution of the interest growth differential



Evolution of the fiscal space/gap



Source: Fiscal Advisory Council (2021)

Table C.2: Parameter overview for Fiscal Sustainability Report 2021

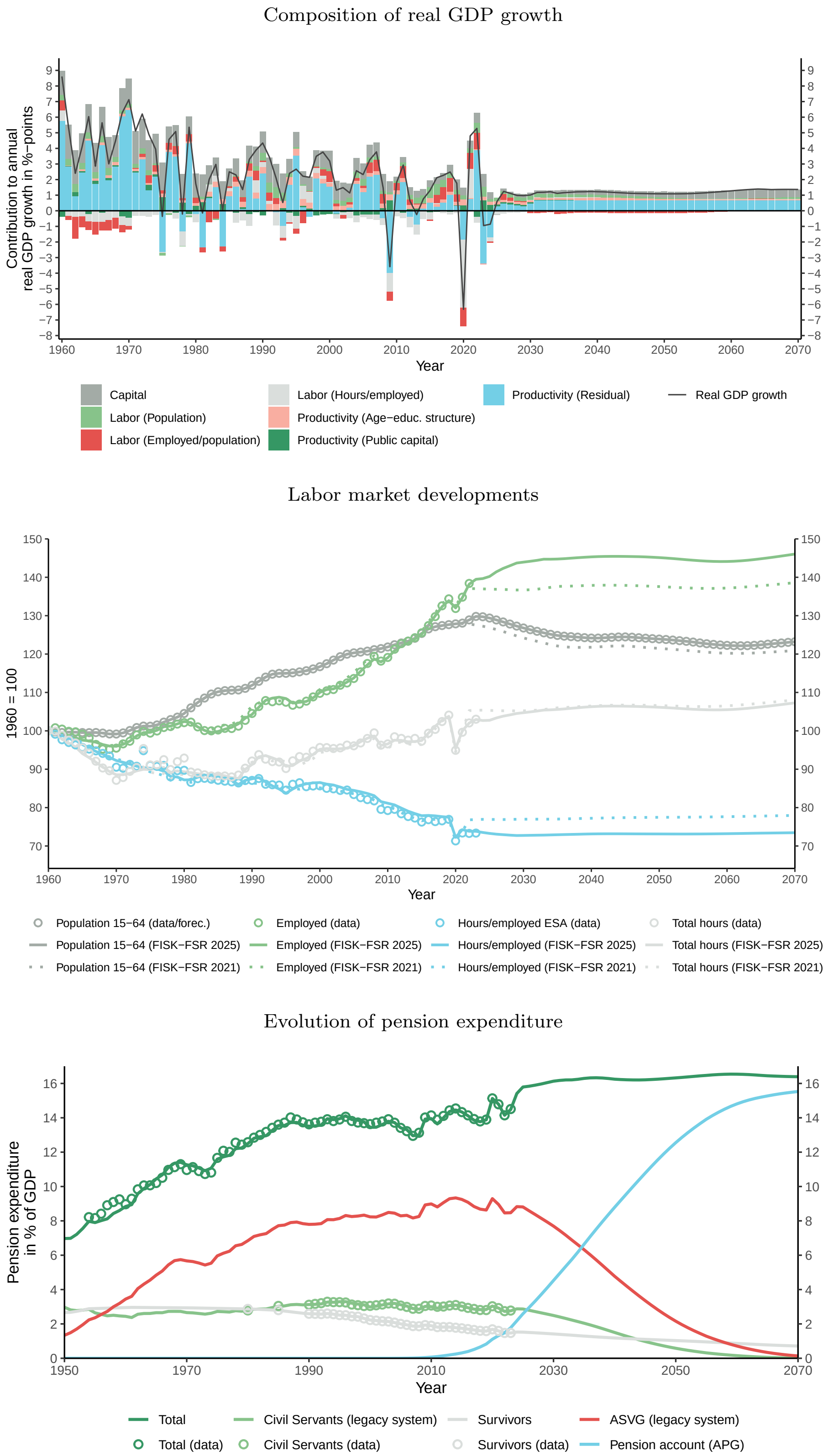
parameter	symbol	value	target/source
number of age groups	$\overline{a}+1$	101	modeling choice
productivity growth	g	time-dep.	match GDP/capita (HMFDA)[1]
discount rate	$(1-\beta)/\beta$	-0.03	targeted asset-to-GDP ratio of about 4
depreciation rate	δ^K	0.049	national accounts (STAT)
depreciation rate (publ. capital)	δ^G	0.049	same as private K
elasticity of intertemporal substitution	σ	0.98	estimated
elasticity of hours[2]	ε^ℓ	0.2	Müllbacher et al. (2017)[3]
elasticity of search effort[2]	ε^s	0.2	Müllbacher et al. (2017)[3]
semi-elasticity of participation[2]	ε^δ	0.1	Müllbacher et al. (2017)[3]
degree of habit formation	κ	0.0	deactivated
retirement disutility elast.	υ_1	1.2	Sánchez-Romero et al. (2024)
household size weight (mult.)	ω_0	0.75	NTA consumption profile
household size weight (power)	ω_1	0.75	NTA consumption profile
elasticity of subst. capital/labor[4]	$1/(1-\epsilon)$	0.75	estimated
elasticity of subst. labor skills	$1/(1-\epsilon^L)$	∞	deactivated[5]
elasticity of subst. young/old labor	λ^{L^s}	∞	deactivated[5]
elast. of subst. dom./foreign C	λ^C	1.2	Breuss et al. (2009), Ratto et al. (2009)
elast. of subst. dom./foreign C^G	λ^{C^G}	1.2	Breuss et al. (2009), Ratto et al. (2009)
elast. of subst. dom./foreign I	λ^I	1.2	Breuss et al. (2009), Ratto et al. (2009)
elast. of subst. dom./foreign I^G	λ^{I^G}	1.2	Breuss et al. (2009), Ratto et al. (2009)
elast. of subst. dom./foreign A^V	σ^{A^V}	1.2	same as for goods
elast. of subst. dom./foreign A^G	σ^{A^G}	1.2	same as for goods
elast. of subst. V/G assets	σ^A	5.0	Keuschnigg and Dietz (2007)
export price elasticity	λ^E	$-\infty$	deactivated[5]
foreign demand dom. assets A^V elast.	$\lambda^{V,h}$	∞	deactivated
foreign demand dom. assets A^G elast.	$\lambda^{G,h}$	∞	deactivated
import share C	ξ^C	0.30	input-output tables (STAT)
import share C^G	ξ^{C^G}	0.13	input-output tables (STAT)
import share I	ξ^I	0.40	input-output tables (STAT)
import share I^G	ξ^{I^G}	0.12	input-output tables (STAT)
love of variety (monop. comp.)	ϑ	∞	deactivated[5]
efficiency public capital	σ^G	0.082	Bom and Ligthart (2014)
share of constrained households	$1-\pi^s$	$\{0.5, 0.3, 0.1\}$	Coenen et al. (2005), Ratto et al. (2009)
vacancy posting costs	c	0	deactivated[5]
return rate on gov. bonds	r^G	time-dep.	match interest expenditure (HMFDA)[1]
return rate on firm bonds	r^V	time-dep.	match current account position (HMFDA)
age-skill productivity profiles	θ^i	-	EU-SILC
age-skill taxes/transfers profiles	-	-	EU-SILC
labor tax function parameters[6]	-	time-dep.	derived from Reiss and Schuster (2020)
age-skill C^G profiles	$c^{G,i}$	-	Hammer (2015) and estim. trends
fiscal shift parameters	-	-	match revenue/expend. (HMFDA)
age-skill hours profiles	$\varphi_0^{\ell,i}$	-[7]	EU-SILC
age-skill search effort profiles	$\varphi_0^{s,i}$	time-dep.	match job flows acc. to RBLMS (STAT)
age-skill participation profiles	$\varphi_0^{\delta,i}$	-[7]	RBLMS (STAT)
family status transition rates	$\pi^{j=\{s,w,d\},i}$	time-dep.	match pop. by fam. status (HMFDA)
job separation rates	χ^i	time-dep.	match job flows acc. to RBLMS (STAT)
pension eligibility	ϕ^i	time-dep.	match effect. retirement age (HMFDA)

Notes: The calibration values chosen do not necessarily precisely match those of the sources cited, but are reasonably close. 'time-dep.'... time-dependent. 1) For the future based on assumptions. 2) Income effects specification (Section 3.1). 3) The chosen values imply an overall Marshallian elasticity of 0.1 which is at the lower bound of the estimates in Müllbacher and Nagl (2017). 4) Standard CES-production function (Section 3.6.1). 5) Deactivated for more stability of dynamic calibration to the historical data. 6) Excluding labor taxes which are subject to estimated tax functions. 7) Time-independent except for added trends to match historical dispersion of hours and participation.

C.3 Fiscal Sustainability Report 2025

In 2025, the Austrian Fiscal Advisory Council published its second iteration of the Fiscal Sustainability Report (FSR) (Fiscal Advisory Council, 2025). The main analysis was carried out with the FISK OLG Model version 2.3. Differences in the results and conclusions compared to FSR 2021 occurred because of methodological changes and the use of more recent data. The quantitatively most important change in the method (see change log in appendix B) is the inclusion of a climate module, as described in section 6.1. The deep parameters, as reported in table C.2, were left unchanged. The model was adjusted to replicate the medium-term macroeconomic forecast by WIFO from January 2025 and a technical update of the medium-term fiscal forecast by FISK from December 2024. Long-run labor productivity growth was revised downward to 1.0% p.a., based on the long-run historic average. The elasticities of substitution between different forms of energy were set between those in Reiter (2024) and Varga et al. (2021): $\sigma^{VA} = 0.75$, $\sigma^{K} = 0.5$, $\sigma^{EK} = 2.0$, $\sigma^{EKC} = 0.4$, $\sigma^{EKD} = 0.5$, $\sigma^{EKD1} = 0.4$, $\sigma^{EKD2} = 0.4$, and $\sigma^{EKD3} = 0.4$, see figure 6.1. Relative efficiency coefficients were set to replicate those implicitly assumed in the long-term energy consumption projections by Krutzler et al. (2023) used in the Austrian National Energy and Climate Plan from 2024. The considerably worse current fiscal stance of Austria has led to a substantially deteriorated outlook on the long-run development of the primary balance. The fiscal gap already in the short-run exceeds 2% of GDP and is expected to increase to 7% of GDP by 2070. The inclusion of climate- and energy-related effects on the budget additionally contributed to a more pessimistic outlook by widening the fiscal gap in 2070 by an additional 1.3% of GDP. The most important channels here are the drop in revenue from (brown) energy-related taxes and non-compliance costs related to the European Effort Sharing Regulation (ESR). Figures C.5 to C.7 exemplarily show some of the results of the analysis.

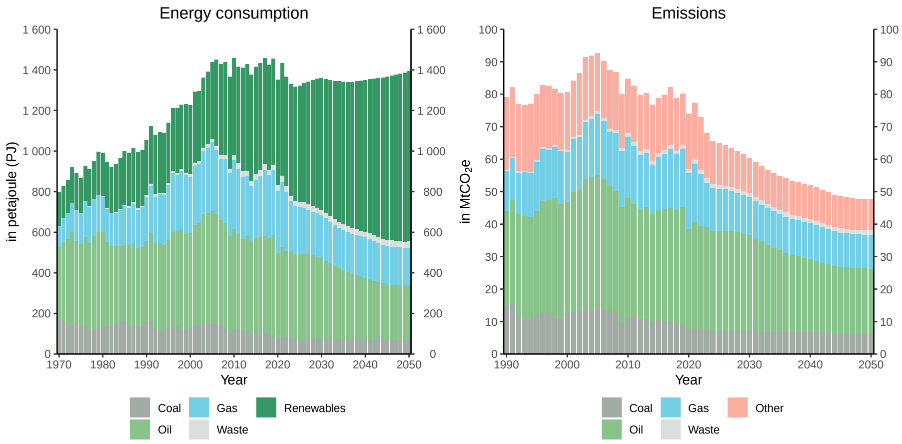
Figure C.5: Main results of FSR 2025 for Austria 3/3



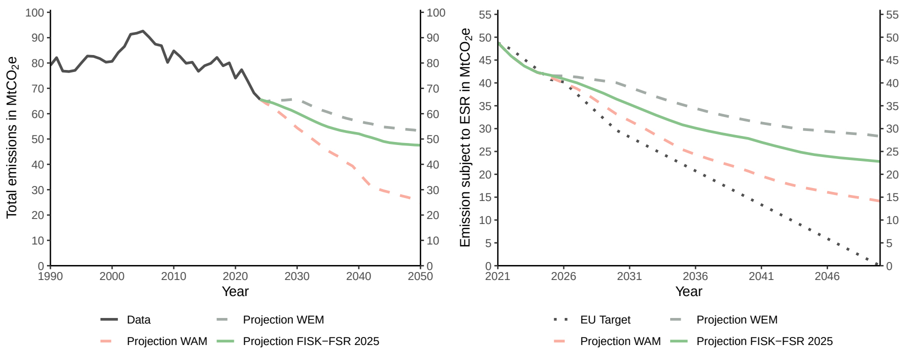
Source: Fiscal Advisory Council (2025)

Figure C.6: Main results of FSR 2025 for Austria 2/3

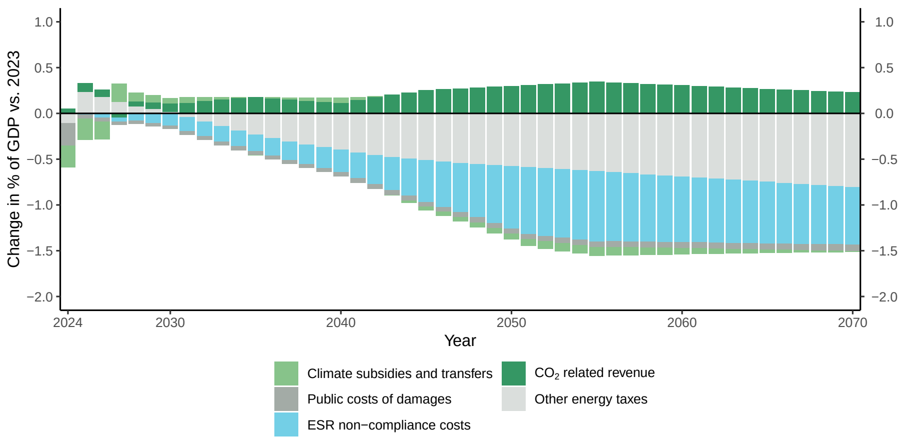
Evolution of energy consumption and GHG emissions



GHG emission projections (total and ESR only)



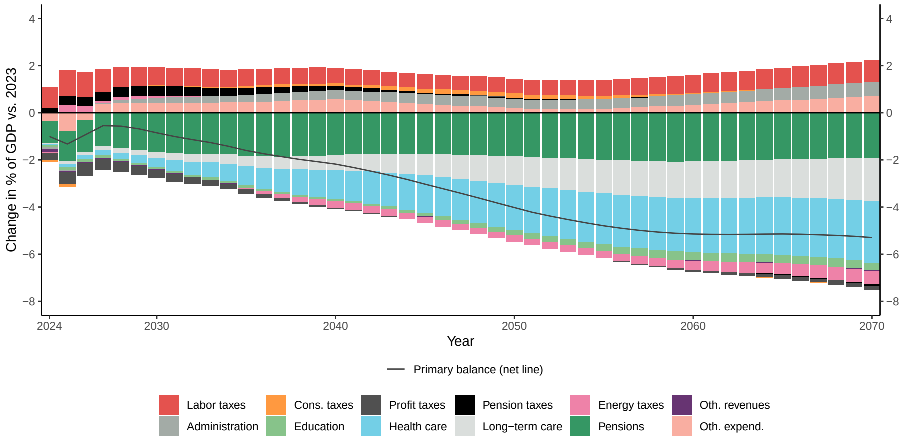
Change in climate- and energy-related budget items



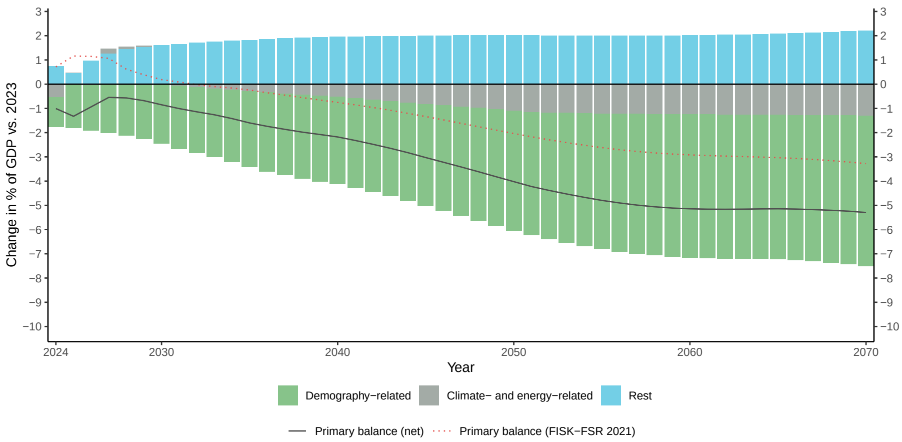
Source: Fiscal Advisory Council (2025)

Figure C.7: Main results of FSR 2025 for Austria 3/3

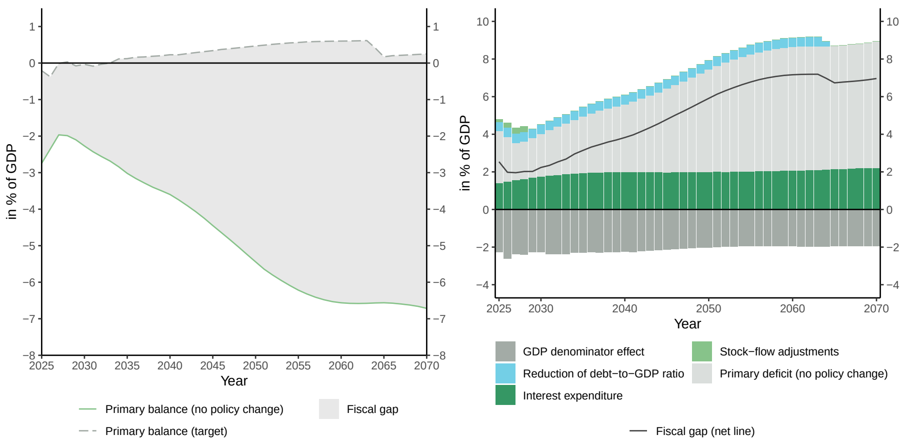
Evolution of the primary balance by item



Evolution of the primary balance by topic



Evolution of the fiscal gap



Source: Fiscal Advisory Council (2025)

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